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Visualization techniques for proofs: Implications for enhancing conceptualization and understanding in mathematical analysis

Jonatan Muzangwa [1], Ugorji Ogbonnaya [2*]

[1] Great Zimbabwe University, Masvingo, Zimbabwe
[2] University of Pretoria, Pretoria, South Africa

[*] Correspondence: ugorji.ogbonnaya@up.ac.za

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Abstract

Visual images are frequently utilized to elucidate concepts in general mathematics and geometry; however, their application in mathematical analysis remains uncommon. This paper demonstrates how visual imagery can enhance the proof of certain theorems in mathematical analysis. It emphasizes the importance of visualization in the learning and understanding of mathematical concepts, particularly within mathematical analysis, where diagrams are seldom employed. The paper focuses on the reasoning processes used by mathematicians in proving selected fundamental theorems of mathematical analysis. It provides illustrative examples where visual images are instrumental in performing specific subtasks within proof development and in completing the proofs. The proofs discussed include the sum of the first n natural numbers, the sum rule of integration, the mean value theorem for derivatives, the mean value theorem for integrals, and Young's Inequality. This paper underscores that visual images serve not only as persuasive tools but also as bridges between symbolic representations and real-world understanding.

Keywords: definite integrals, imagery, mathematical analysis, mean value theorem, visualisation

Introduction

The field of mathematics is rich with visual relationships inherent in its concepts, ideas, and methods, which can be intuitively represented in various ways. Recognizing the crucial role of visual thinking and visualization in mathematics learning, educators often integrate visual representations, such as images or diagrams, into the teaching of mathematical concepts. This approach involves understanding the importance of students' visualization processes, their abilities, and the pedagogical strategies designed to enhance instruction by establishing visual learning environments (Gates, 2018; Giaquinto, 2011; Makina, 2010).

In contrast to the abstract nature typically associated with the logico-deductive formal proofs used by mathematicians, research has shown that visualization plays a central role in the creative, exploratory, and proof processes that lead to new results. An illustrative example is the anecdote of Norbert Wiener, who resolved a complex proof with the assistance of enigmatic pictures, highlighting the power of visual thinking in mathematics (Guzman, 2002). Similarly, some scholars have successfully employed visual representations to prove various mathematical theorems, such as the area of a circle (Browne, 2022), the Pythagorean theorem (Foo et al., 1999; Santos & Quaresma, 2010), and the sum of the first *n* odd numbers (Relaford-Doyle et al., 2017). Researchers like Ahmad (2021), Arcavi (2003), Giaquinto (2011), Guncaga et al. (2019), Guzman (2002), and Svitek et al. (2022) have emphasized the importance of visualization and visual reasoning in the process of learning mathematics. Visual images can significantly enhance the understanding of complex mathematical concepts by rendering abstract ideas more concrete and accessible (Coessens et al., 2021; Quinnell, 2022; Žakelj & Klancar, 2022). Furthermore, they serve as valuable tools or strategies for solving mathematical problems (Barbosa & Vale, 2021; Kaitera & Harmoinen, 2022; Parame-Decin, 2023) and facilitate creativity in mathematical problem-solving (Bicer et al., 2023; Kell et al., 2013; Vale & Barbosa, 2023). Stylianou and Silver (2004) found that expert problem solvers utilize visual representations "as dynamic objects to explore the problem space qualitatively, develop a better understanding of the problem situation, and guide their solution planning and execution of problem-solving activity" (p. 353).

However, Stylianou and Silver (2004) also noted that the use of visual representations presents challenges for many students in mathematical problem-solving, as visual images can be complex collections of ideas that may not be easily accessible to all learners. This difficulty may stem from students' limited exposure to visual images in mathematics instruction. Such challenges often result in a disconnect between the formal aspects of mathematical analysis and the underlying meanings they are intended to convey (Eisenberg & Dreyfus, 1991). In his study "Adding Structure to the Transition Process to Advanced Mathematical Activity," Engelbrecht (2010) explored the challenges faced by undergraduate students as they transition from calculus to mathematical analysis. He observed that this shift can be particularly difficult, sometimes even traumatic, and advocated for a methodological approach that integrates visualization techniques with symbolic reasoning to enhance conceptual understanding and mitigate these difficulties.

Although numerous studies have demonstrated the critical role of visualization in the teaching and learning of various mathematical concepts, the majority of this research has focused on general mathematics and geometry. There appears to be a scarcity of literature on the role of visualization in learning mathematical analysis. Mathematical analysis, typically taught after calculus at the undergraduate level, relies on a rigorous logical foundation grounded in the axioms of real numbers. It emphasizes the importance of understanding proofs and the reasoning behind specific steps. Grouws (1992) suggested that organizing mathematical knowledge around key components such as definitions, lemmas, propositions, theorems, examples, and counterexamples is essential for students to effectively comprehend and prove mathematical statements. This structured approach facilitates understanding, logical reasoning, and the development of proficiency in mathematical problem-solving.

At the undergraduate level, mathematical thinking predominantly relies on the definitions of mathematical terms rather than the use of visual images. Nurwahyu and Tinungki (2020) elucidated the relationship between concept image and concept definition, highlighting the importance of visualization in forming mental representations of complex mathematical concepts. Visual elements contribute significantly to intuitive reasoning, which can be categorized into diagrammatic reasoning, analogical reasoning, and the use of prototypes (Gates, 2018; Presmeg, 2020).

There is a prevailing belief that visualization in mathematical analysis is heuristic rather than a means of discovery or proof. Barwise and Etchemendy (2019) noted that "despite the obvious importance of visual images in human cognitive activities, visual representation remains a second-class citizen in both the theory and practice of mathematics. In particular, we are all taught to look askance at proofs that make crucial use of diagrams, graphs, or other non-linguistic forms of representation" (p. 160). This paper argues that certain theorems may be better understood when accompanied by diagrams. As Barker-Plummer and Bailin (1997) succinctly stated, "visualization distinguishes following a proof from seeing it to be true" (p. 26). We present five examples where visual images can be instrumental in facilitating the proofs of mathematical analysis theorems. Demeke (2016) contended that while multiple proofs may exist for a given mathematical theorem, the choice of a particular proof should be based on its value in a given context. The author emphasized that "mathematicians may value a proof for reasons such as comprehensibility, explanatory power, and originality" (p. 11). Visual images can help to clarify proofs and enhance their comprehension.

In the following sections, we discuss the role of imagery in mathematics and visual thinking in learning mathematical analysis concepts. It was followed by an overview of the conceptual research method and a discussion on how visual representations can be employed to enhance the understanding of five fundamental mathematical analysis concepts.

Imagery in Mathematics

Imagery is a cognitive process that involves the mental representation of entities not currently perceived by the senses (Goldstein, 2011; Sternberg, 2009). This process includes the creation and manipulation of mental images, which facilitate various cognitive functions such as memory, problem-solving, and comprehension. Within this context, the focus is on visual imagery, specifically the capacity to "see" in the absence of a visual stimulus. Gauss famously argued that in mathematical argumentation, the scaffolding used to construct a proof should be concealed, leaving only the final product visible. In contrast, Hadamard (1945) emphasized the importance of informal reasoning, which involves thinking without words, relying on visual imagery and mental images that may not initially be expressible in language. This type of reasoning also includes the exploration of ideas through activities akin to piecing together a puzzle.

Mathematical learning is guided by two fundamental principles: inductive and deductive reasoning. Inductive reasoning involves deriving general principles from specific cases, while deductive reasoning works in the opposite direction, applying general principles to arrive at specific conclusions. The incorporation of imagery in mathematical learning is particularly aligned with inductive reasoning. Mason (2002) explains the purpose of assigning learning tasks in mathematics goes beyond merely obtaining correct answers. It also involves fostering an understanding of the broader applicability of various methods. As Mason (2002) suggests, generality emerges within the cognitive realm and is communicated through verbal descriptions, diagrams, and symbolic representations in the observable realm. Recognizing generality requires moving beyond specific instances to grasp overarching concepts, a process facilitated by descriptions that evoke mental imagery, as well as diagrams that are perceived as frames in a complex, film-like manner of processing mathematical information.

Visual thinking is essential in understanding calculus concepts, a field that evolved to address quantitative physical problems beyond the scope of geometry and arithmetic. Differentiation and integration, the two core components of calculus, were developed to determine the gradient of a curve and to calculate the area of irregular shapes, respectively. Visual representations, often in the form of diagrams, play a critical role in aiding the comprehension of these concepts. The development of calculus is attributed to Newton and Leibniz, who approached its fundamental concepts differently—Newton through a geometric lens and Leibniz through an analytical perspective. At the undergraduate level, calculus encompasses topics such as real numbers, sequences, functions, limits, continuity, differentiation, and integration. Although diagrams are frequently employed to clarify concepts, proofs in calculus rarely rely on them, as much of the discipline involves the manipulation of functions (Miller, 2012).

Visual Thinking in Learning Mathematical Analysis Concepts

Mathematical analysis seeks to rigorously formalize and precisely articulate the intuitive concepts underlying calculus. While intuition offers a direct perception of truth or facilitates immediate reasoning, there are instances where the complexity of mathematical concepts necessitates more than intuition alone (Giaquinto, 2011; Hanna, 1991; Tall, 1991). In such cases, formal proof and validation become indispensable.

The notion that visual thinking is often viewed with skepticism in the realm of mathematical analysis is particularly noteworthy. Visualization can play a significant role in various aspects, such as aiding in the comprehension of formulas, serving as a reminder of counterexamples, and inspiring ideas for proofs. However, the inherent limitation lies in the fact that visual thinking, by itself, may not always ensure or preserve truth in the rigorous context of mathematical analysis.

Methods

This paper employs a conceptual research method, as outlined by Gilson and Goldberg (2015). The conceptual research method relies on the examination and synthesis of existing academic literature to explore and integrate diverse ideas and concepts related to a specific subject (Medvedeva et al., 2021). Rather than collecting and analyzing empirical data, this approach involves the analytical exploration and discussion of theoretical ideas, often using illustrative examples to support the arguments. It is a scientific method that advances intellectual discourse by synthesizing and integrating concepts from existing literature (Kraus et al., 2022).

In this paper, visual images are presented as tools to facilitate the proof of five mathematical analysis theorems, which are often challenging for students. These theorems include the sum of the first n natural numbers, the sum rule of integration (a key property of definite integrals), the mean value theorem for derivatives, the mean value theorem for integrals, and Young's Inequality.

Results and Discussion

Theorem 1: The sum of the first n natural number: $1+2+3+\ldots+n = \frac{n(n+1)}{2}$**.**

Understanding proof by induction is often challenging for students, largely due to its heavily theoretical presentation (Demeke, 2016). However, this theorem can be both proved and better understood through the use of visual representations. This paper leverages the concept of triangular numbers, as discussed by Giaquinto (2011), to support this argument. Triangular numbers serve as a prime example, representing the sum of the first *k* positive integers for any positive integer *k*. The first triangular number is 1, the second is 1 + 2, and, in general, the kth triangular number is represented as $1+2+3+\ldots+(k-1)+k = T(k)$.

The objective of the proof is to establish a formula that enables the computation of the sum of the nth triangular number. Figure 1 presents both numerical and visual algebraic representations of this concept, illustrated using colored dots. T(n) represents the total number of dots, and the arrangement should visually resemble a triangle.

n	1	2	3	4	5	…	k
T(n) as a sum	1	1+2	1+2+3	1+2+3+4	1+2+3+4+5		1+2+…+k
T(n) as a triangle						…	…
T(n)	1	3	6	10	15	…	1+2+… +k

Figure 1. Visual illustration of triangular numbers

One can observe that $T(k) = 1+2+3+\ldots+(k-1)+k$. Figure 2 is the visual representation of aligning two triangles with the same number of dots but different colours to form a rectangle.

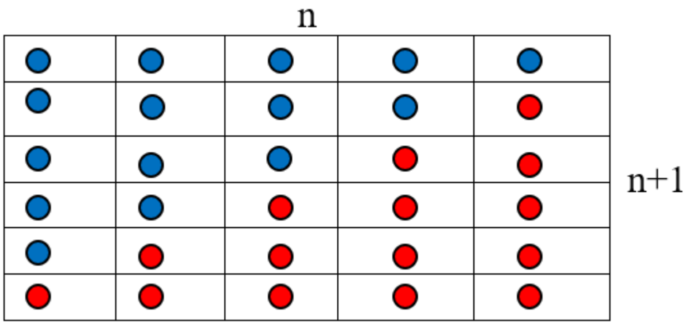

Figure 2. Visual illustration of T(k).

One may observe that T(blue dots) = T(red dots). Hence the total number of dots in the rectangle is known to be the number of dots lengthwise multiplied by the number of dots width wise (*l x w*).

2T(n) = T(blue dots) + T(red dots) = Total number of dots of the rectangle = *n(n+1)* .

Therefore $T(n) = \frac{n(n+1)}{2}$ as required.

Theorem 2. Visual proof of the sum rule of integration: $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)\, dx$.

The sum rule of integration is one of the properties of definite integrals. The following is the proof of the properties as found in Stewart (2008). The property states that if *f* and *g* are continuous functions in [a, b] then,

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

The visual presentation is shown in Figure 3.

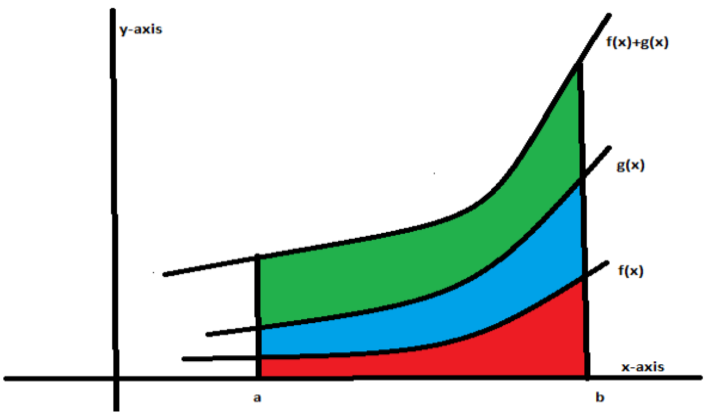

Figure 3. Visual diagram of the sum of two functions

The sum rule of integration asserts that the integral of a sum is equal to the sum of the integrals. Specifically, for positive functions, this rule implies that the area under the curve of $f + g$ equals the sum of the areas under the curves of f and g. Figure 3 illustrates this concept, aiding in the comprehension of the rule's validity. In the figure, the area under $f + g$ is represented by RED, BLUE, and GREEN. The area under f is depicted as RED, while the area under g is shown as RED and BLUE. Visually, the rule demonstrates that the area represented by RED, BLUE, and GREEN is equivalent to the sum of the areas represented by [RED+BLUE] and [RED]. This indicates that if $f(x) \leq g(x)$ for values of x in the interval [a,b], then the GREEN portion is always equal to the RED portion for positive functions.

Proof

$$\int_a^b [f(x) + g(x)]dx = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\Delta x$$

$$= \lim_{n\to\infty} [\textstyle\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x]$$

$$= \lim_{n\to\infty} \textstyle\sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \textstyle\int_a^b f(x)dx + \int_a^b g(x)dx, \text{ as required.}$$

Theorem 3. The Mean Value Theorem for Derivatives

The primary method for visually representing a real-valued function is through the construction of its graph. As Stewart (2008) emphasizes, the graph effectively encapsulates the behavior or "life history" of the function. This graphical representation typically manifests as a line on the Cartesian plane (x-y plane), which may vary in form—ranging from straight and curved to jagged and discontinuous. A continuous function is depicted as an uninterrupted line, which can be traced without lifting the pencil. In contrast, a differentiable function is represented by a line that not only remains unbroken but also exhibits perceptual smoothness, free from abrupt changes in direction or jagged segments.

The gradient of a function at any given point is visually represented by the ratio of the vertical distance to the horizontal distance. Utilizing these visual representations, one can illustrate the steps leading to the proof of the Mean Value Theorem for derivatives. The theorem is articulated as follows:

Let f be a function that satisfies the following hypotheses:
f is continuous on the closed interval [a,b].
f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that f'(c) = (f(b)-f(a))/(b-a) or equivalently f'(c) (b-a) =f(b)-f(a).

Visual Representation

Before proving the theorem, it can be visually represented and interpreted. Figure 4 shows that the gradient (slope) of the line joining points (a, f(a)) and (b,f(b)) can be written as m = (f(b)-f(a))/(b-a) which is the same expression as on the right of the Mean Value Theorem. Since f'(c) is the gradient of the tangent line at the point (c, f(c)), the Mean Value Theorem in the form given above says that there is at least one point P(c, f(C)) on the graph where the gradient of the tangent line is the same as the gradient of the secant line joining (a,f(a)) and (b, f(b)).

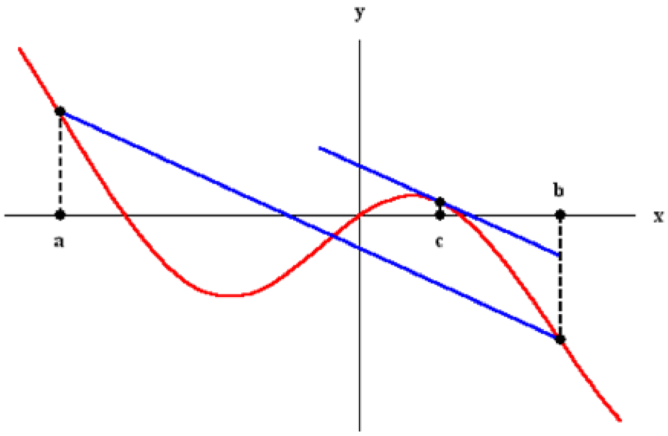

Figure 4. Visual diagram for mean value theorem for derivatives

Proof

Apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line defined above (Stewart, 2008). The equation of the secant line can be written as $\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$ or as y = f(a) + $\frac{f(b)-f(a)}{b-a}(x-a)$. So, as shown in Figure 5.

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Firstly, it must be verified that h satisfies the three conditions required for Rolle's Theorem.

The function h is continuous on [a,b] since h is a polynomial and hence differentiable.

In fact, $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

$h(a) = f(a)-f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$ and $h(b) = f(b)-f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0$

Therefore h(a) = h(b). Since h satisfies the condition for Rolle's Theorem that there is a number c in (a,b) such that h/(c) = 0. Therefore

$0 = h'(c) = f'(c) = -\frac{f(b)-f(a)}{b-a}$ to result in $f'(c) = -\frac{f(b)-f(a)}{b-a}$

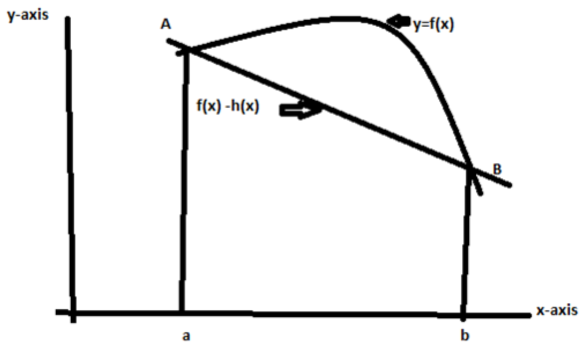

Figure 5. Visual illustration of the relationship between f(x), h(x), and equation of the secant line

Comments

The challenge frequently arises when students encounter the introduction of the function *h(x)* in a proof presented in a refined manner. Students may question the purpose of including such a function. However, the clarity provided by the illustration in Figure 4 helps to address this difficulty effectively.

The diagrams above vividly illustrate the effectiveness of visual reasoning in comprehending concepts in mathematical analysis. The proof progresses by tracing the logical steps backward from the visual representation to the formulation of a formal mathematical statement.

Theorem 4. The Mean Value Theorem for Integrals

Visual arguments alone are rarely reliable as pathways to discovery. Nevertheless, visualization plays a critical role in analytic discovery by frequently stimulating ideas for proofs (Giaquinto, 2011). The visual representation of the Mean Value Theorem for integrals further illustrates how analytical reasoning can emerge from visual insights. The theorem is stated as follows:

If f is continuous on [a,b] then there is a point c in (a,b) such that $\int_a^b f(x)dx = f(c)(b-a)$.

Visual Representation

Figure 6 provides a visual representation of the theorem's statement. $\int_a^b f(x)dx$ represents the area under the curve y= f(x) from a to b. The area under the curve limited between the lines x=a and x=b is YELLOW+GREEN+ BLUE. The right-hand part of the theorem [f(c)(b-a)] represents the area of the rectangle whose width is (b-a) and length f(c) which visually is YELLOW + GREEN + RED. Without losing generality one can observe that YELLOW+GREEN+BLUE = YELLOW + GREEN +RED which implies that for some c in the interval (a,b) the RED part is always equal to the BLUE part. The proof then is going to guarantee the existence of this c.

Figure 6. Visual illustration of the Mean Value Theorem for Integrals

Proof

Since f is continuous on [a,b], it is integrable on [a,b]. Moreover since [a,b] is closed and bounded, it means that f attains its maximum and minimum values M and m, respectively on [a,b]. In other words, there are points x_0 and x_1 in [a,b] such that M= f(x_0) and m = f(x_1), see Figure 7 for a visual representation of the Mean Value Theorem for Integrals (Stewart, 2008).

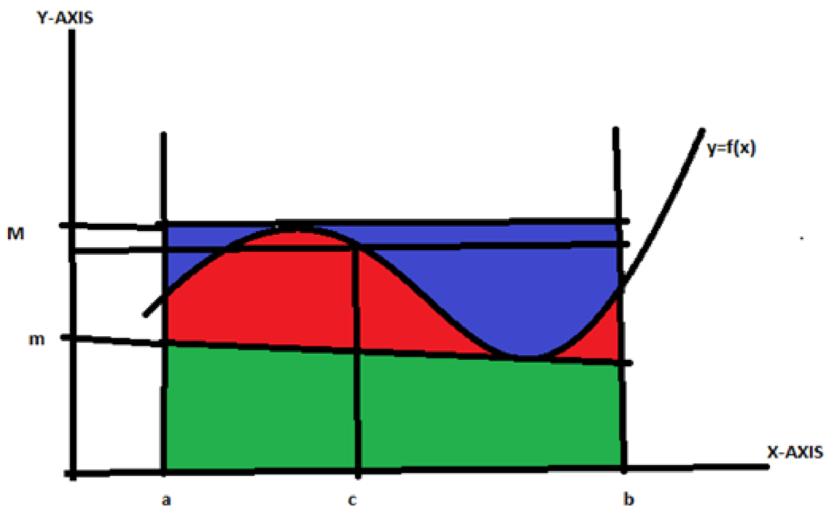

Figure 7. Mean Value Theorem for Integrals visual representation**.**

Thus m$\leq f(x) \leq M$ which implies m(b-a) $\leq \int_a^b f(x)dx \leq M(b-a)$ ……………x

Therefore m $\leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$.

Now let$\mu = \frac{1}{b-a}\int_a^b f(x)dx$, implies that m$\leq \mu \leq M$. Therefore, there exist c ∈ (a, b) such that f(c) = μ . Thus $\int_a^b f(x)dx = f(c)(b-a)$ as required.

Comment

Note that the key statement for the proof is the one marked ᴥ. This statement is a summary of the visual representation of Figure 4. If A = $\int_a^b f(x)dx$, A1 = m(b-a) and A2 = M(b-a) then clearly A1≤ A ≤A2 and the rest is algebra.

Usually, it is difficult for undergraduate learners to comprehend the above proof if it is not accompanied by the above diagrams.

Theorem 5: Young's Inequality

Young's Inequality is an important auxiliary result used in the derivation of Hölder's Inequality, which is a pivotal step in proving the Minkowski Inequality. Mastery of these inequalities is essential for the study of normed linear spaces, typically covered in postgraduate functional analysis courses. The inclusion of Young's Inequality in this context highlights the crucial role of visualization in enhancing the understanding of mathematical analysis. The lemma is stated as follows:

Let p and q be conjugate exponents, with 1<p, q <∞ and α, β ≥0.

Then $\propto\beta \le \frac{\propto^p}{p} + \frac{\beta^q}{q}$ - Young's inequality (Mitrinovic & Vasic, 1970; Moharana, 2014).

Visual Representation

If p and q are conjugate exponents, then $p - 1 \ = \frac{1}{q-1}$ which implies that if f(t) = tp-1 then f-1(t) = tq-1. This statement is represented in Figure 8.

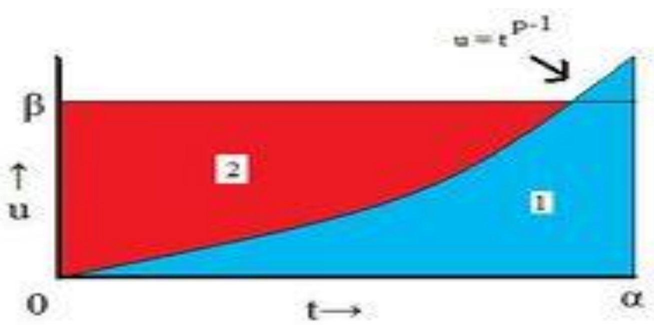

Figure 8. Visual diagram of the idea of the proof of Young's Inequality

Examining the diagram, we can infer that the area of the rectangle, with a width of β and length ∝, is less than or equal to the combined area of the BLUE part and the RED part. This relationship can be expressed as:

$$\propto\beta \le \int_0^\alpha f(t)dt + \int_0^\beta f^{-1}(t)dt = \int_0^\alpha t^{p-1}dt + \int_0^\beta t^{q-1}dt = \frac{\propto^p}{p} + \frac{\beta^q}{q} \text{ as required.}$$

Comment

It can be noted that the key statement for the analytical proof is the statement above represented by ᴥ, f(α) = $\frac{\propto^p}{p} + \frac{\beta^q}{q}$ - $\propto\beta$ which is a deduction of the visual observation in Figure 8. Generally, if the above theorem is proved without the diagram, it presents difficulties, especially in justifying the choice of f(α). The idea of the proof is backward working from imagery to a formal mathematical statement.

The examples provided in this paper emphasize the pivotal role of visualization in proving fundamental theorems in mathematical analysis. In the context of proving theorems, lemmas, and properties, visualization often serves as the initial foundation from which concepts and methodologies are developed. It also acts as an efficient means of conveying mathematical ideas. The examples discussed herein illustrate that while visual images can be instrumental in developing proofs, they are often discarded once formalized.

Guzman (2002) highlights the remarkable utility of visualization not only in the initial process of mathematization but also in the teaching and learning of mathematics. This underscores the importance of developing visual skills and introducing them to individuals new to the subject. This principle extends beyond geometry, where the importance of visual elements is clear, to mathematical analysis. In analysis, ideas, concepts, and methods are rich in visual, intuitive, and geometric content that frequently emerges in the mental processes of analysts.

Although Giaquinto (2011) suggests that visual arguments are rarely reliable for drawing analytic conclusions and are infrequently pathways to discovery, the acknowledgment of visualization's critical role in analytic discovery is significant. While visual thinking is often deemed unreliable for concluding mathematical analysis, due to the nature of concepts that may not be visually representable or defy visuospatial expectations, two examples illustrating how visualization can be misleading are encountered in the construction of graphs for certain functions, such as:

$f(x) = x^2 - 2$, for x real in [1,2], and

$$g(x) = \begin{cases} x^2 - 2, for\ x\ rational\ in\ [1,2] \\ undefined, otherwise \end{cases}$$

The curves of these functions exhibit no distinct visual differences. For any given rational number, there exist rational numbers arbitrarily close on both sides, resulting in an infinite density of rational numbers between any two of them. As a result, the curve of g(x) will not exhibit any noticeable gaps and is visually indistinguishable from the curve of f(x). It can be observed that *g* does not attain a zero value since there is no rational *x* for which $x^2-2=0$, despite the function satisfying Bolzano’s Theorem.

In general, many visual arguments may appear to substantiate the validity of certain propositions in mathematical analysis, sometimes leading to the belief that they constitute proof. However, these visual arguments often only provide a conceptual foundation for a formal proof. Nevertheless, such visual arguments are highly valuable. With increased experience, experts can more readily transition from a visually presented idea to an analytical proof. Thus, visual arguments are of significant utility for experts, aiding in the proof development process and enhancing comprehension within the field.

Conclusion

Changing representation registers can be pivotal in enhancing mathematical understanding. This paper illustrates how visual representations can facilitate students' access to mathematical knowledge, as evidenced by examples such as the proofs of the sum of the first n natural numbers, the sum rule of integration, the Mean Value Theorem for derivatives, and the Mean Value Theorem for integrals. Consequently, we advocate for the integration of visual representations in the teaching of mathematical analysis proofs whenever feasible. This approach can render proofs more accessible to students and foster a deeper comprehension of mathematical concepts.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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