Journal of Honai Math, 9(2), 225-244, August 2026
https:/doi.org/10.30862/jhm.v9i2.1250

Refining a PMRI-Based Learning Trajectory for Angle Measurement Through a Rotation-Based Bridging Phase

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Received: 8 April 2026 | Revised: 26 May 2026 | Accepted: 31 July 2026 | Published: 25 August 2026

Abstract

This study investigates students' learning trajectory in understanding angle measurement within a Pendidikan Matematika Realistik Indonesia (PMRI)-based instructional design through a design research approach. It aims to examine how students progressively construct conceptual understanding of angles, moving from informal contextual experiences toward formal measurement using a protractor, with particular attention to transitions across multiple representations. Data were collected through a sequence of teaching experiments involving contextual tasks based on the traditional Limas house, non-standard circular representations, and formal protractor-based measurement. Students' written work, classroom observations, and responses were analyzed to compare the actual learning trajectory with the Hypothetical Learning Trajectory (HLT). The findings reveal that students initially interpreted angles through visual perception, emphasizing shape-related features rather than rotational meaning. Although circular representation activities promoted an emerging understanding of angles as parts of a complete rotation, this conceptual development remained fragmented. A critical discontinuity occurred during protractor use, where persistent difficulties in alignment, scale selection, and interpreting angle magnitude hindered conceptual progression. To address these challenges, a bridging phase integrating circular representations, rotational movement, and degree measurement was introduced. This intervention enabled students to reconceptualize angles as measures of rotation, resulting in greater measurement accuracy and conceptual consistency, particularly for benchmark angles of 45°, 90°, and 180°. The study concludes that learning angle measurement is a non-linear process requiring explicit representational mediation, with the bridging phase serving as a crucial instructional mechanism for facilitating students' transition from informal experiences to formal mathematical reasoning.

Keywords: PMRI; Angle Measurement; Learning Trajectory; Bridging Phase; Rotational Reasoning; Design Research

Introduction

Angle measurement constitutes a foundational concept in geometry, underpinning students' spatial reasoning, geometric thinking, and broader mathematical reasoning. Developing a robust understanding of angles enables students to interpret geometric relationships, reason quantitatively about rotation, and engage with increasingly sophisticated mathematical ideas. Nevertheless, many students continue to experience persistent conceptual difficulties in interpreting and measuring angles accurately (Amalia et al., 2024). This challenge remains particularly significant within the Indonesian educational context, where recent curriculum reforms emphasize mathematical reasoning, problem solving, and assessment competence as essential learning outcomes aligned with national education standards (Kemendikdasmen, 2025; UUD RI Tahun 1945, 2016). Despite these policy expectations, students' achievement in geometry continues to be constrained by fragile conceptual foundations, suggesting that current instructional practices have not yet sufficiently supported meaningful conceptual development.

A substantial body of research has demonstrated that students commonly interpret angles as static geometric figures rather than as measures of rotational change (Smith et al., 2014). This misconception frequently manifests in procedural errors during angle measurement, including incorrect alignment of the protractor, confusion between the inner and outer scales, and the belief that angle magnitude depends on the length of its rays instead of the amount of rotation they represent (Bustang et al., 2013; Sinclair & Bruce, 2015). Such findings indicate that procedural proficiency alone is insufficient for developing a coherent understanding of angle measurement. Furthermore, these difficulties are closely associated with limitations in spatial reasoning, visual thinking, and students' ability to coordinate multiple mathematical representations in geometry (Yaniartini et al., 2019; Zhou et al., 2022).

To address these challenges, previous studies have explored a variety of instructional approaches, including the use of manipulatives, contextual learning environments, and dynamic representations to support students' understanding of angles (Seah & Horne, 2021; Simpson et al., 2024). Although these studies have reported positive effects on learning outcomes, they have primarily evaluated the effectiveness of particular instructional interventions rather than examining how students' conceptual understanding develops throughout the learning process. Similarly, existing research on learning trajectories generally assumes a relatively smooth progression from informal experiences toward formal angle measurement using a protractor. Consequently, comparatively little attention has been devoted to the conceptual discontinuities that emerge during this transition or to the mechanisms through which students coordinate multiple representations while constructing increasingly formal understandings of angle.

This limitation represents an important theoretical and pedagogical gap. Developing a meaningful understanding of angle measurement requires students to coordinate contextual experiences, visual representations, rotational movement, circular models, and formal measurement tools within a coherent conceptual framework. Previous studies have shown that students' geometric reasoning progresses through increasingly sophisticated levels of understanding; however, transitions between these levels are rarely linear and often involve substantial cognitive challenges (Vojkuvkova, 2012; Zhou et al., 2022). From this perspective, representational mediation may play a critical role in maintaining conceptual continuity as students move from intuitive interpretations toward formal mathematical reasoning. Nevertheless, the role of representational mediation within students' learning trajectories of angle measurement remains insufficiently understood. Addressing this gap requires investigating not only students' final learning outcomes but also the developmental processes through which conceptual understanding is progressively constructed across successive instructional phases.

The present study addresses this issue through the Indonesian Realistic Mathematics Education known as PMRI approach by employing the cultural context of the Limas traditional house. Grounded in the principles of guided reinvention and progressive mathematization, PMRI provides meaningful and culturally relevant learning experiences that enable students to construct mathematical ideas from informal experiences toward increasingly formal reasoning (Zulkardi & Putri, 2019; Eerde, 2013). Context-based learning has consistently been shown to enhance students' mathematical engagement and conceptual development by connecting mathematical ideas with their lived experiences (Nurniqta et al., 2025; Nalurita et al., 2010). Moreover, contextual instructional media, including comics, e-comics, and culturally situated learning materials, have demonstrated considerable potential for strengthening conceptual understanding and engagement in mathematics learning (Kaur, 2019). Within this study, the Limas traditional house functions not merely as a contextual illustration but as an authentic mathematical context through which students progressively reconstruct the concept of angle from everyday spatial experiences.

Building upon this instructional context, the study focuses on the critical transition that occurs when students move from informal recognition of angles toward formal measurement using a protractor. Particular attention is given to a rotation-based bridging phase designed to support students in reconceptualizing angles as quantified measures of rotation rather than as static visual forms. The interpretation of students' conceptual development is informed by APOS theory, which provides a theoretical lens for examining how learners progressively construct, reorganize, and internalize mathematical concepts through successive cognitive structures. Integrating APOS theory with PMRI enables the study to explain not only what students learn but also how their conceptual understanding evolves during the learning process.

This study contributes to the growing body of PMRI-based learning trajectory research by proposing a refined Hypothetical Learning Trajectory (HLT) that explicitly incorporates representational mediation as a mechanism for supporting conceptual continuity between informal and formal understandings of angle measurement. Rather than conceptualizing learning as a linear progression, the proposed trajectory identifies critical transition points where conceptual discontinuities emerge and demonstrates how an explicitly designed bridging phase facilitates students' movement toward formal mathematical reasoning. Consequently, the study contributes both theoretically, by extending current understanding of learning trajectories in geometry through the integration of representational mediation and APOS theory, and practically, by providing an empirically grounded instructional framework that can guide the design of geometry learning environments supporting students' conceptual development.

Accordingly, this study aims to investigate how students construct conceptual understanding of angle measurement across successive instructional phases within a PMRI-based learning trajectory. Specifically, the study addresses the following research questions:

1. How do students develop conceptual understanding of angle measurement as they progress from informal contextual experiences to formal protractor-based measurement within a PMRI learning trajectory?
2. What critical conceptual transition points emerge during students' movement from informal angle recognition to formal angle measurement?
3. How does a rotation-based bridging phase facilitate students' coordination of multiple representations and support conceptual continuity during the development of angle measurement?

Methods

This study employed a design research methodology to develop, implement, and refine a learning trajectory for angle measurement in elementary mathematics. Design research was selected because it enables the systematic development of local instructional theory through iterative cycles of instructional design, classroom implementation, and retrospective analysis, while maintaining a close connection between theoretical conjectures and students' actual learning processes (Gravemeijer & Cobb, 2006; Plomp & Nieveen, 2013). Rather than merely evaluating instructional effectiveness, design research seeks to explain how learning develops and how instructional designs can be progressively refined based on empirical evidence generated in classroom settings.

Following the framework proposed by Gravemeijer and Cobb (2006), the study comprised three interconnected phases: preliminary design, teaching experiment, and retrospective analysis. To strengthen the feasibility of the instructional sequence, a pilot experiment was conducted prior to the main teaching experiment as part of the iterative design process.

Preliminary Design

During the preliminary design phase, an initial HLT was constructed by synthesizing previous research on angle measurement, learning trajectories, and the principles of PMRI. The HLT specified the intended learning goals, instructional activities, and conjectures regarding students' anticipated progression from informal contextual reasoning toward formal angle measurement using a protractor.

The instructional sequence was designed around the cultural context of the Limas traditional house, selected through the principle of didactical phenomenology to provide students with meaningful opportunities to explore angle concepts within authentic situations (Gravemeijer & Cobb, 2006). The initial HLT assumed that students would progressively interpret angles as measures of rotational space, establish conceptual connections between contextual representations and formal mathematical representations, and ultimately transition smoothly to accurate protractor use.

Pilot Experiment

Before the main teaching experiment, a pilot study was conducted to evaluate the feasibility and practicality of the proposed instructional sequence. This phase focused on examining the clarity of learning tasks, students' accessibility to the contextual activities, and the appropriateness of the instructional timing. Classroom observations during the pilot experiment revealed several practical issues, particularly students' uncertainty in identifying angle arms and locating the vertex within contextual representations. Based on these findings, minor revisions were made to the wording, sequencing, and visual presentation of several instructional tasks before classroom implementation.

Teaching Experiment

The teaching experiment constituted the primary phase of classroom implementation, during which the revised HLT was enacted in an authentic learning environment. The instructional activities were designed to elicit students' mathematical reasoning, representational strategies, and conceptual development as they progressed from contextual exploration toward formal angle measurement.

Throughout the teaching experiment, the researcher conducted systematic classroom observations while maintaining sufficient flexibility to make minor instructional adjustments when necessary. Such adaptations were intended to preserve the integrity of the learning trajectory while responding to students' emerging reasoning during classroom interactions. Consequently, the teaching experiment served not only to implement the instructional design but also to document the evolution of students' mathematical thinking across successive learning activities.

Retrospective Analysis

The retrospective analysis constituted the principal analytical phase of the study. Data collected from classroom observations, students' written work, and semi-structured interviews were analyzed by systematically comparing the conjectured learning processes described in the HLT with students' actual Learning Trajectories (LT). Consistent with the principles of design research, this comparison enabled the researchers to examine the extent to which the instructional conjectures were supported by empirical classroom evidence and to identify necessary refinements to the instructional theory (Cobb et al., 2003).

Particular attention was given to identifying mismatches between the anticipated and observed learning processes, conceptual discontinuities, and critical transition points that emerged as students moved from informal representations of angles toward formal measurement using a protractor. These findings informed subsequent revisions of the learning trajectory and contributed to the refinement of the local instructional theory underlying the study.

Implementation Timeline

The study was conducted over seven weeks, consisting of a one-week pilot experiment, a four-week teaching experiment comprising three instructional meetings per week, and a two-week retrospective analysis conducted concurrently with data organization and interpretation presented in Table 1.

Table 1. Implementation timeline

Phase	Duration	Frequency
Pilot Experiment	1 week	2 meetings
Teaching Experiment	4 weeks	3 meetings/week
Retrospective Analysis	2 weeks	Continuous analysis

Evolution of the HLT

An essential characteristic of design research is the iterative refinement of instructional theory based on empirical classroom evidence. Accordingly, the initial HLT was continuously evaluated and revised throughout both the pilot and teaching experiments by comparing the conjectured learning processes with students observed learning trajectories.

Initially, the HLT assumed that students would progressively interpret angles as measures of rotational space, establish meaningful connections between contextual representations and formal mathematical concepts, and transition smoothly to protractor-based measurement. However, classroom observations revealed several important discrepancies between these theoretical conjectures and students' actual learning processes. Rather than conceptualizing angles as measures of rotation, many students interpreted them primarily as static geometric figures. Furthermore, although students were generally able to manipulate the protractor procedurally, they frequently lacked an understanding of the underlying measurement principles. Persistent difficulties were observed in aligning the protractor baseline with the angle arms, selecting the appropriate measurement scale, and interpreting angle magnitude as the amount of rotation rather than as a visual property of the figure. In addition, evidence of spontaneous rotational reasoning was rarely observed during classroom activities.

These findings suggested that contextual experiences alone were insufficient to support students' transition from informal reasoning to formal angle measurement. Consequently, the HLT was refined through the introduction of an explicit bridging phase that systematically connected contextual situations, circular representations, rotational movement, and formal protractor-based measurement. Additional instructional supports, including guided alignment activities, tracing tasks, and structured reflection questions, were incorporated to strengthen students' coordination of multiple representations and facilitate conceptual continuity throughout the learning trajectory.

The evolution of the HLT from its initial assumptions to the revised instructional design is summarized in Table 2.

Table 2. Evolution of the HLT

Initial HLT Assumption	Empirical Finding	Instructional Revision
Students transition smoothly to protractor use.	Students relied on procedural manipulation of the protractor without conceptual understanding.	Introduced an explicit bridging phase connecting rotational reasoning to formal measurement.
Students naturally interpret angles as measures of rotation.	Students predominantly perceived angles as static geometric shapes.	Incorporated contextual tasks emphasizing rotational movement through the Limas traditional house context.
Students can directly interpret angle magnitude using a protractor.	Students experienced difficulties aligning the baseline and selecting the appropriate measurement scale.	Added guided alignment, tracing, and measurement activities before formal protractor use.
Contextual experiences alone support conceptual transfer.	Contextual understanding did not consistently transfer to formal measurement.	Strengthened explicit reflection activities to connect contextual experiences with formal mathematical concepts.

Participants and Context

The participants were 29 fourth-grade students from a public elementary school in Palembang, Indonesia. The school was selected to represent a typical urban public-school context in which students had previously studied introductory geometric concepts but had limited experience with formal angle measurement using a protractor. All instructional activities were conducted during regular mathematics lessons in collaboration with the classroom teacher to preserve the authenticity of the classroom environment.

Data Collection

Multiple sources of qualitative data were collected throughout the teaching experiment to document students' conceptual development during the learning process. These included classroom observations, students' worksheets, and semi-structured interviews. Semi-structured interviews were conducted with purposively selected students representing diverse patterns of mathematical thinking observed during classroom activities. The selection included students who demonstrated conceptual misconceptions, procedural difficulties in using the protractor, and varying levels of representational understanding. This purposive sampling enabled the researchers to examine a broad range of learning trajectories and to explore how different students negotiated the transition from informal to formal understandings of angle measurement.

Data Analysis

Data analysis was conducted iteratively throughout the retrospective analysis phase following the principles of design research. Classroom observations, students' written work, and interview transcripts were analyzed by systematically comparing the HLT with students' actual LT, thereby examining the extent to which the instructional conjectures were supported by empirical classroom evidence (Gravemeijer & Cobb, 2006).

The analysis employed iterative thematic coding to identify patterns in students' mathematical reasoning across multiple data sources. The coding focused on four analytical dimensions: (1) students' interpretation of mathematical representations, (2) procedural use of the protractor, (3) evidence of rotational reasoning, and (4) conceptual discontinuities occurring during transitions between instructional phases. Emerging themes were continuously compared across classroom observations, students' written responses, and interview data to identify recurring learning patterns and critical transition points within the learning trajectory.

Particular analytical attention was devoted to moments in which students shifted from informal contextual representations toward formal protractor-based measurement. Rather than treating procedural errors as isolated mistakes, recurring inconsistencies in students' explanations and measurement strategies were interpreted as indicators of underlying structural learning challenges. These analyses enabled the identification of critical transition points and informed the refinement of the learning trajectory through the introduction of the bridging phase.

To enhance the credibility of the findings, triangulation was conducted across the three data sources, and the coding process was reviewed through peer debriefing among the research team. Iterative comparisons between the conjectured HLT and the observed LT further strengthened the trustworthiness of the analysis by ensuring that instructional refinements were grounded in systematic empirical evidence rather than post-hoc interpretation (Cobb et al., 2003).

Results

This section presents the empirical findings of the teaching experiment by tracing students' actual learning trajectory (ALT) in developing an understanding of angle measurement. Consistent with the design research methodology, the analysis compares students observed learning processes with the conjectured HLT to identify critical moments of conceptual development and instructional refinement. The findings are organized into four successive phases of the learning trajectory: (1) initial informal understanding of angles, (2) emergence of qualitative angle comparison, (3) the critical transition point during formal protractor use, and (4) conceptual reorganization through representational mediation.

Initial Informal Understanding of Angles

The learning trajectory began with students exploring angle concepts through the contextual setting of the traditional Limas house. At this stage, students identified angles primarily by referring to visually salient features of the building, such as roof edges, corners, and intersecting structural components. Their reasoning was largely perceptual, relying on the appearance of geometric forms rather than on the mathematical meaning of angle as a measure of rotation.

Analysis of students' written responses and classroom discussions revealed three recurring patterns. First, students tended to perceive angles as static visual objects rather than as measures of rotational change. Second, the classification of angle types, including acute, right, and obtuse angles, was frequently inconsistent, indicating that students had not yet established stable conceptual criteria for distinguishing different angle categories. Third, students' judgments were predominantly influenced by perceptual salience, with visually prominent shapes receiving greater attention than the underlying geometric relationships.

These findings suggest that students entered the instructional sequence with an intuitive but largely visual conception of angle. Although the contextual activities successfully encouraged students to identify geometric features within authentic situations, they did not yet support an understanding of angles as quantitative measures of rotation. Consequently, students' initial reasoning reflected perceptual interpretation rather than formal mathematical conceptualization.

Emergence of Qualitative Angle Comparison

Following the contextual exploration, students were introduced to non-standard representations using circular paper models to compare angle magnitude. This instructional phase was intended to support the transition from purely visual recognition toward reasoning about angles as portions of a complete rotation.

As shown in Figure 1, students used circular paper to represent angles as fractions of a full circle, allowing them to compare angle sizes qualitatively through relationships such as halves and quarters. For several students, these representations provided an important conceptual bridge between everyday visual experiences and more formal geometric reasoning.

Figure 1. Students' Non-standard Representations of Angle Magnitude

The analysis identified three prominent learning patterns during this phase. First, some students successfully associated one-quarter of a circle with a right angle, demonstrating an emerging understanding of angle magnitude in relation to a complete rotation. Second, other students continued to treat the circular models primarily as visual aids without establishing an explicit conceptual connection between the represented sector and rotational movement. Third, although qualitative comparison of angle size became increasingly evident, students' reasoning remained inconsistent and context dependent.

These findings indicate that the circular representations promoted an initial shift from perceptual recognition toward quantitative reasoning. Nevertheless, students had not yet fully coordinated the visual, spatial, and rotational meanings of angle. Instead, the circular models functioned as intermediate representations that partially supported conceptual development while revealing the need for further instructional mediation before formal measurement could be meaningfully introduced.

Critical Transition Point During Formal Protractor Use

The most significant conceptual difficulty emerged when students were introduced to formal angle measurement using a protractor. Although many students demonstrated increasing confidence during the preceding activities, this instructional phase revealed a substantial discontinuity between their informal representational understanding and the conceptual demands of formal measurement.

Classroom observations, students' worksheets, and interview data consistently identified three major sources of difficulty. First, many students incorrectly aligned the center of the protractor with the vertex of the angle. Second, students frequently failed to position the baseline of the protractor along one of the angle rays before attempting to read the measurement. Third, considerable confusion arose when students selected between the inner and outer numerical scales, often resulting in incorrect measurements despite otherwise appropriate procedural actions.

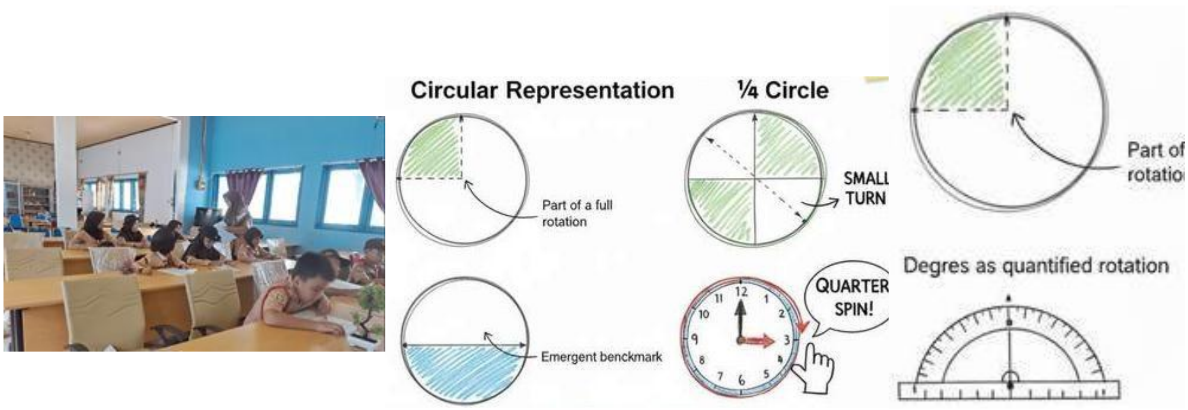
Importantly, these difficulties were not limited to students who had experienced earlier conceptual challenges. Several students who successfully completed the contextual and representational activities nevertheless encountered substantial difficulties during protractor use. This finding suggests that successful engagement with informal representations did not automatically transfer to formal measurement procedures.

Rather than representing isolated procedural errors, these recurring difficulties indicate a structural discontinuity within the learning trajectory. Specifically, students appeared to lack an explicit conceptual understanding of the protractor as a tool for quantifying rotational movement. Consequently, formal measurement introduced cognitive demands that could not be resolved through procedural instruction alone, thereby revealing a critical transition point between informal representational understanding and formal mathematical measurement.

Conceptual Reorganization through the Bridging Phase

To address the conceptual discontinuity identified during formal protractor use, an explicit bridging phase was introduced to connect three complementary representations of angle: circular representations, rotational movement, and degree measurement. The instructional sequence was designed to support students in reconceptualizing angles as quantified measures of rotation rather than as static geometric forms.

As illustrated in Figure 2, students were guided to relate portions of a complete circle to rotational movement and corresponding degree values before using the protractor as a formal measurement tool. The conceptual model illustrating the representational mediation introduced during the bridging phase, in which circular representations and rotational movement support students' interpretation of the protractor as a tool for measuring rotation. Through this sequence of activities, the protractor was gradually introduced not merely as an instrument for reading numerical values but as a representation of rotational measurement.



[Circular Representation] → [Rotational Movement] → [Protractor as Rotation Tool]

Figure 2. Bridging Phase Linking Circular Representations, Rotational Movement, and Degree Measurement

Evidence from classroom observations, students' worksheets, and interview data indicates that the bridging phase substantially supported students' conceptual development. Following the intervention, students demonstrated more accurate alignment of the protractor with the angle vertex and baseline, selected the appropriate measurement scale with greater consistency, and measured benchmark angles such as 45°, 90°, and 180° with noticeably improved accuracy. Moreover, students' verbal explanations shifted from describing angles as visual shapes toward interpreting them as "turns" or "rotations," indicating a fundamental change in their conceptualization of angle measurement.

This conceptual reorganization became particularly evident when students physically coordinated circular models with rotational movement before translating these relationships into degree measures. Rather than relying on procedural manipulation of the protractor, students increasingly interpreted angle measurement as the quantification of rotational change. These findings suggest that conceptual understanding emerged through the coordination of multiple representations, with the bridging phase functioning as an essential mechanism for connecting informal reasoning with formal mathematical measurement.

Refinement of the Learning Trajectory: Comparing the HLT and the ALT

Comparison between the initial HLT and students' ALT revealed that learning angle measurement was considerably more complex than initially anticipated. Whereas the original HLT assumed a relatively continuous progression from contextual exploration to formal protractor use, the empirical findings demonstrated a substantial conceptual discontinuity during the transition to formal measurement.

The observed learning trajectory was characterized by five successive phases: contextual recognition, qualitative angle comparison, critical transition during formal measurement, bridging through representational mediation, and conceptual consolidation. This sequence indicates that students did not progress directly from informal reasoning to formal measurement. Instead, successful conceptual development depended upon an intermediate phase in which multiple representations were explicitly coordinated before the protractor could be interpreted as a meaningful mathematical tool.

The HLT–ALT comparison further demonstrated that the initial instructional design underestimated the cognitive demands associated with interpreting the protractor as a representation of rotational measurement. Although contextual activities enabled students to recognize and compare angles qualitatively, these experiences alone were insufficient to support conceptual transfer to formal measurement. Consequently, the introduction of the bridging phase emerged as a necessary instructional refinement rather than a supplementary learning activity.

The refined learning trajectory therefore extends the original HLT by explicitly incorporating representational mediation as a structural component of angle learning. Rather than conceptualizing learning as a linear progression, the revised trajectory acknowledges the existence of a critical transition point and demonstrates that students require explicit opportunities to coordinate contextual experiences, circular representations, rotational movement, and formal measurement before achieving conceptual consolidation. These findings provide empirical support for refining PMRI-based learning trajectories by positioning the rotation-based bridging phase as an essential mechanism for facilitating students' transition from intuitive understandings toward formal mathematical reasoning.

Overview of the ALT

The overall pattern of students' conceptual development is summarized in Figure 3, which synthesizes the ALT identified through the teaching experiment. The diagram illustrates that students' progression toward formal angle measurement was not linear but instead involved a critical transition point that required explicit instructional mediation before conceptual understanding could be consolidated.

Contextual Recognition

(Angles in Limas House)
↓
Qualitative Comparison
(Non-standard representations)
↓
⚠ Critical Transition Point ⚠
(Breakdown in conceptual continuity)
↓
Bridging Phase
(Rotation-based mediation)
↓
Conceptual Consolidation
(Angle as measured rotation)

Figure 3. Narrative Representation of the ALT

As illustrated in Figure 3, students initially developed meaningful contextual understandings of angles and gradually coordinated qualitative comparisons through circular representations. However, these achievements did not automatically transfer to formal angle measurement. Instead, students encountered a distinct conceptual discontinuity when required to interpret the protractor as a tool for measuring rotational change. This finding demonstrates that successful performance during earlier instructional phases was insufficient to support formal measurement without additional representational mediation.

Comparison between the initial HLT and the observed ALT revealed that the original instructional design underestimated the cognitive complexity associated with interpreting angle measurement as quantified rotation. Consequently, the bridging phase emerged as a necessary instructional refinement rather than a supplementary activity. By explicitly coordinating contextual experiences, circular representations, rotational movement, and protractor use, the bridging phase enabled students to reorganize their understanding and establish meaningful connections between informal and formal mathematical representations.

To summarize the principal conceptual difficulties identified throughout the learning trajectory, Table 3 presents the recurring misconceptions observed during classroom implementation together with the instructional responses incorporated into the refined learning trajectory.

Table 3. Empirically Identified Conceptual Difficulties and Corresponding Instructional Responses

Conceptual Difficulty	**Empirical Evidence**	**Instructional Response**
Angles interpreted according to side length or visual appearance	Students' worksheet responses during the initial contextual activities	Contextual comparison tasks using the *Limas*house and visual contrast activities

Angles perceived as static geometric objects rather than measures of rotation	Interview responses and inconsistent angle classifications	Circular representations emphasizing rotational movement
Procedural use of the protractor without conceptual understanding	Classroom observations during formal measurement activities	Rotation-based bridging phase connecting circular representations to protractor use
Confusion between the inner and outer measurement scales	Inconsistent angle measurements across tasks	Explicit comparison of measurement scales in relation to rotation direction
Incorrect alignment of the protractor with the angle vertex and baseline	Students' worksheets and classroom observations	Guided alignment and tracing activities with structured scaffolding

Finally, these findings demonstrate that conceptual development in angle measurement depends not only on contextual experiences or procedural instruction but also on explicit representational mediation that supports students in interpreting angles as quantified measures of rotation. The refined learning trajectory therefore provides an empirically grounded instructional model for facilitating students' transition from intuitive visual reasoning toward formal mathematical understanding.

Discussion

The findings suggest that students' difficulties in angle measurement are more appropriately understood as a breakdown in the coordination of multiple representations than merely as procedural limitations in using a protractor. Rather than following a linear progression from contextual experiences to formal mathematical measurement, students' learning trajectory was characterized by a critical transition point at which previously successful informal reasoning no longer supported conceptual understanding of formal measurement. This transition required students to coordinate visual, embodied, and symbolic representations of angle, revealing that conceptual development depends not only on procedural competence but also on representational integration.

From Visual Perception to Rotational Reasoning

The results indicate that students initially conceptualized angles as static visual objects derived from the perceptual characteristics of contextual shapes. This finding is consistent with previous studies reporting that learners frequently interpret angles according to their visual appearance rather than as measures of rotational change (Sinclair & Bruce, 2015; Smith et al., 2014). Although contextual activities and circular representations encouraged students to compare angle magnitude qualitatively, these representations alone were insufficient to transform their underlying conception of angle.

The introduction of the protractor fundamentally altered the representational demands of the learning situation. Students were required to coordinate spatial orientation, rotational direction, and numerical measurement simultaneously, yet many continued to interpret the protractor procedurally by focusing on reading numerical scales rather than understanding its conceptual role as a representation of rotation. This finding aligns with previous research describing the complex relationship between conceptual and procedural knowledge in mathematics learning (Latsi & Kynigos, 2022; Rittle-Johnson et al., 2001). Consequently, the observed learning difficulties are better interpreted as failures of representational coordination than as isolated procedural errors.

Viewed through the lens of APOS theory, students' conceptual development appeared to progress from Action, in which they identified angles through observable contextual features, to Process, where circular representations encouraged reasoning about rotational movement. However, many students encountered substantial difficulty transforming this process into an Object, namely understanding angle as a measurable quantity represented by the protractor. The introduction of the bridging phase subsequently supported the construction of a coherent Schema by integrating contextual experiences, rotational reasoning, and formal measurement into a unified conceptual structure. This interpretation suggests that conceptual difficulties emerged not because students failed to perform procedures, but because they had not yet reorganized their cognitive structures sufficiently to coordinate multiple representations of angle.

Representational Coordination as a Mechanism of Conceptual Change

A central interpretation of the findings concerns the epistemological obstacle underlying students' understanding of angle. Students' prior conception of angle as a static geometric figure conflicted with the formal mathematical definition of angle as a quantified measure of rotation. This conflict became particularly evident at the critical transition point, where students were required to shift from perceptual reasoning toward rotational quantification. Existing visual schemas frequently interfered with this transition, resulting in persistent misalignment of the protractor, confusion between measurement scales, and inconsistent interpretations of angle magnitude.

These findings support the argument that conceptual change in geometry is constrained by deeply rooted epistemological commitments rather than by instructional exposure alone (Rittle-Johnson et al., 2001; Seah & Horne, 2021). More importantly, they suggest that overcoming such obstacles requires students to coordinate multiple forms of representation rather than merely acquiring additional procedural skills. In this study, representational coordination emerged as the principal cognitive mechanism through which conceptual change occurred.

The bridging phase therefore functioned not simply as an instructional scaffold but as a representational mediation mechanism connecting contextual experiences, circular representations, embodied rotational movement, and symbolic degree measurement. This interpretation extends previous research demonstrating that manipulatives are effective only when explicitly connected to underlying conceptual structures (Clements & Burns, 2000; Rich, 2021). Rather than serving merely as visual aids, the circular models became transitional representations through which students reconstructed the meaning of angle as quantified rotation. The empirical evidence showing improvements in protractor alignment, scale selection, and explanations of angle magnitude further supports the view that successful learning depended on students' ability to coordinate these complementary representations.

Refining Learning Trajectory Theory through Representational Mediation

The findings also contribute to the refinement of learning trajectory theory within geometry education. Existing learning trajectory models generally describe students' progression from informal experiences toward formal mathematical understanding as a sequence of increasingly sophisticated reasoning processes (Gravemeijer & Cobb, 2006; Lesh & Lehrer, 2003; Zulkardi et al., 2020). The present study suggests that such progression should not necessarily be regarded as continuous. Instead, conceptual development may involve critical transition points where existing representations are insufficient to support further learning.

The comparison between the HLT and the ALT demonstrated that the original instructional design underestimated the representational demands associated with formal angle measurement. The emergence of the bridging phase therefore represents more than an instructional modification; it constitutes a refinement of the underlying local instructional theory. Specifically, the findings indicate that representational mediation is a structural component of successful learning trajectories rather than an optional pedagogical enhancement.

Accordingly, the present study proposes that learning trajectories in geometry should be conceptualized not only as sequences of progressively more sophisticated mathematical activities but also as trajectories of representational coordination. From this perspective, conceptual progress depends upon learners' ability to integrate contextual, visual, embodied, and symbolic representations across critical transition points. This theoretical refinement extends existing design research by positioning representational coordination as a central mechanism through which conceptual continuity is established during mathematical learning.

Implications for PMRI and Geometry Instruction

The findings have important implications for the design of PMRI-based geometry instruction. The results suggest that introducing formal measurement tools directly is insufficient for developing meaningful conceptual understanding. Instead, instructional sequences should explicitly support students in constructing rotational reasoning before introducing formal measurement procedures.

Within the PMRI framework, contextual experiences should therefore be complemented by carefully designed representational mediation that progressively links everyday situations, circular representations, embodied movement, and formal mathematical tools. Such sequencing enables students to interpret the protractor as a representation of quantified rotation rather than as an instrument for mechanically reading numerical values.

These findings reinforce previous studies emphasizing the importance of representation-rich learning environments in mathematics education (Apsari et al., 2020; Bustang et al., 2013; Sinclair & Bruce, 2015). More importantly, however, they extend this body of research by identifying both the specific stage at which representational failure occurs, and the instructional mechanism required to overcome it. Consequently, the bridging phase should not be regarded merely as an additional instructional activity but as an essential component of learning designs intended to facilitate students' transition from intuitive visual reasoning toward formal mathematical understanding.

Conclusion

This study investigated students' learning trajectory in angle measurement within a PMRI-based instructional design and identified a critical transition point during the movement from informal contextual reasoning and circular representations to formal protractor-based measurement. The findings indicate that students' difficulties at this stage cannot be adequately explained as procedural limitations alone. Rather, they reflect a representational and epistemological discontinuity in which students struggle to reconceptualize angles from static visual forms to quantified measures of rotation. This transition highlights the complexity of conceptual development in geometry and demonstrates that successful engagement with informal representations does not necessarily transfer to formal mathematical measurement.

The principal contribution of this study lies in the refinement of the HLT for angle measurement. Whereas the initial HLT assumed a relatively continuous progression from contextual experiences to formal measurement, the empirical findings revealed the necessity of an intermediate bridging phase that explicitly coordinates contextual experiences, circular representations, rotational movement, and protractor-based measurement. The refined learning trajectory therefore extends existing PMRI-based learning trajectory research by demonstrating that representational mediation constitutes a structural component of conceptual development rather than an optional instructional support. More broadly, the study suggests that learning trajectories in geometry should be understood not only as sequences of increasingly sophisticated mathematical activities but also as processes of representational coordination through which learners progressively integrate visual, embodied, and symbolic forms of mathematical reasoning.

From a pedagogical perspective, the findings underscore the importance of preparing students conceptually before introducing formal measurement tools. Rather than presenting the protractor as an isolated procedural instrument, instruction should incorporate learning experiences that explicitly connect rotational movement, circular representations, and degree measurement, thereby enabling students to interpret angle as a quantified measure of rotation. Such representational mediation provides an effective foundation for supporting meaningful conceptual understanding and reducing persistent misconceptions in angle measurement.

This study is subject to several limitations. The proposed learning trajectory was developed and evaluated within a single instructional context involving one group of fourth-grade students in an Indonesian elementary school. Consequently, caution should be exercised when generalizing the findings to other educational settings or learner populations. Future research could examine the applicability of the refined learning trajectory across different mathematical domains involving measurement, geometry, and spatial reasoning, while also investigating how representational mediation may be adapted to accommodate diverse learners and varying instructional contexts. Such studies would further clarify the role of critical transition points in supporting conceptual development across mathematics education.

Acknowledgment

The author expresses gratitude to the Doctoral Program in Mathematics Education at Sriwijaya University for academic support during this study. The author also sincerely thanks SD Negeri 81 Palembang as the research site for providing access and support during the teaching experiment.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, ethical issues, including plagiarism, infringement, fabrication and/or falsification of data, multiple publications and/or submissions, and redundancy have been fully assumed by the authors.

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