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Mapping cognitive transitions in extended-angle trigonometry: A mixed-methods PCA analysis of university students' representational thinking

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Abstract

Many university students experience difficulties connecting graphical, symbolic, and analytical representations when solving extended-angle trigonometric problems, resulting in fragmented conceptual understanding. Previous studies have mainly examined misconceptions or instructional media separately, while limited research has explored how students transition between multiple mathematical representations. This study employed an explanatory sequential mixed-methods approach integrating quantitative and qualitative analyses to investigate students' representational thinking patterns in extended-angle trigonometry learning. Students' mean scores increased significantly from 54.30 to 75.80 ($t = -8.92$, $p < 0.001$), indicating substantial improvement in conceptual understanding after the instructional intervention. The findings revealed diverse thinking patterns, with most students relying on visual representations, while others preferred symbolic procedures. Students demonstrating analytical flexibility showed deeper representational coordination and conceptual comprehension, enabling smoother transitions among visualization, symbolic manipulation, and analytical reasoning. Principal Component Analysis (PCA) was employed for dimensionality reduction and visualization of students' cognitive tendencies, while thematic analysis of interviews and think-aloud protocols provided deeper insights into students' problem-solving processes. The study contributes a cognitive-transition framework explaining how students coordinate visualization, symbolic manipulation, and analytical reasoning in trigonometric problem solving and provides implications for designing adaptive instructional strategies that support representational flexibility and conceptual understanding.

Keywords: Cognitive strategies, extended angle trigonometric functions, mathematical thinking patterns, principal component analysis, think-aloud protocol.

Introduction

A robust understanding of trigonometric functions involving extended angles, particularly those exceeding, constitutes a fundamental prerequisite for success in advanced university mathematics, including calculus and related analytical disciplines (Alajmi & Al-Kandari, 2022; Zengin, 2021). From a conceptual perspective, students are expected to develop integrated understandings of trigonometric relationships through numerical, graphical, and analytical representations (Mkhwanazi et al., 2023; Nordlander, 2022). Such representational competence enables learners to identify underlying mathematical structures, establish meaningful connections among concepts, and cultivate flexible mathematical thinking. Consequently, effective trigonometry instruction should extend beyond procedural memorization and facilitate students' comprehension of how trigonometric functions behave across different quadrants and coordinate systems (Aylward & Cronjé, 2022; Brieger et al., 2020). Students who demonstrate sophisticated mathematical thinking are generally capable of transferring these conceptual understandings across mathematical domains and interdisciplinary contexts. Moreover, technological resources, including mathematical modeling software and interactive simulations, have been shown to support conceptual development by enhancing visualization, exploratory reasoning, and critical mathematical inquiry.

Despite these expectations, many university students continue to encounter substantial difficulties when connecting symbolic expressions with their corresponding graphical and analytical interpretations in problems involving extended-angle trigonometric functions. Existing research indicates that students frequently rely on memorized formulas and procedural rules without developing a comprehensive understanding of how trigonometric functions operate across multiple quadrants and angle transformations (Teófilo De Sousa et al., 2022). Deficiencies in mathematical reasoning, particularly in interpreting trigonometric transformations beyond (90°), often contribute to persistent misconceptions concerning function signs, angle equivalence, periodicity, and graphical behavior. Furthermore, the limited integration of visual representations and interactive learning environments may constrain students' mathematical intuition and representational flexibility, resulting in fragmented and disconnected conceptual structures (Goldin, 2020; Kondratieva, 2021). Collectively, these findings reveal a significant discrepancy between the conceptual reasoning competencies expected in tertiary mathematics education and the actual cognitive experiences of many learners.

Previous studies have predominantly concentrated on either diagnosing students' misconceptions in trigonometry or evaluating the effectiveness of technology-enhanced instructional interventions. For example, several investigations have explored the use of dynamic graphing tools, interactive simulations, and visualization-oriented learning environments to improve students' understanding of trigonometric concepts (Çekmez, 2020;

Lee et al., 2022). Other studies have focused on identifying common procedural errors, such as inaccuracies in determining trigonometric signs, reference angles, and equivalent-angle relationships. Although these studies have generated valuable insights, the existing literature remains relatively fragmented because misconceptions and instructional interventions are often examined as separate phenomena. Consequently, limited attention has been devoted to understanding how students construct conceptual meaning through coordinated interactions among multiple mathematical representations during problem-solving processes (Hamzah et al., 2021). In particular, little is known about how learners dynamically navigate among visualization, symbolic manipulation, and analytical reasoning when solving trigonometric problems involving extended angles.

Addressing this gap, the present study investigates students' mathematical thinking patterns in conceptualizing extended-angle trigonometric functions through three interconnected representational dimensions: visualization, symbolic manipulation, and analytical reasoning. Rather than focusing exclusively on students' final solutions, this research examines the cognitive processes underlying their reasoning in order to understand how representational thinking influences conceptual understanding and problem-solving strategies. In contrast to previous studies that have examined misconceptions or instructional approaches independently, this study integrates Principal Component Analysis (PCA)-based cognitive mapping, think-aloud protocols, and mixed-methods triangulation to investigate students' transitions across multiple representational forms during trigonometric reasoning. This integrated analytical framework provides a more nuanced understanding of how learners coordinate graphical representations, algebraic transformations, and logical inference while engaging with extended-angle trigonometric tasks.

This study contributes to mathematics education research by positioning representational flexibility as a central mechanism underlying conceptual understanding in trigonometry. By examining the dynamic transitions among visualization, symbolic manipulation, and analytical reasoning, the study extends existing scholarship on mathematical thinking and offers a more comprehensive account of students' cognitive processes in higher mathematics learning. Furthermore, the findings are expected to inform the design of adaptive instructional approaches that integrate interactive simulations, mathematical modeling, and exploratory learning experiences. Such approaches may strengthen students' conceptual understanding, enhance cognitive flexibility, and support more meaningful engagement with trigonometric concepts at the university level.

Methods

This study employed an explanatory sequential mixed-methods design, whereby quantitative findings were subsequently elaborated and interpreted through qualitative evidence derived from semi-structured interviews and think-aloud protocols (Dawadi et al., 2021; Wipulanusat et al., 2020). The integration of quantitative and qualitative strands occurred during the interpretation phase through methodological triangulation, enabling statistical patterns to be examined alongside evidence of students' cognitive processes. Such a design was considered appropriate for investigating how students coordinate and transition among visualization, symbolic manipulation, and analytical reasoning while conceptualizing trigonometric functions involving extended angles. Rather than focusing exclusively on students' final solutions, the study sought to examine the representational and cognitive processes underlying the development of mathematical understanding throughout learning activities.

The quantitative component adopted a one-group pretest–posttest design in which participants completed a pretest prior to the instructional intervention and a posttest upon completion of the intervention (Knapp, 2016; Marsden & Torgerson, 2012). This design enabled the measurement of changes in students' conceptual understanding of trigonometric functions involving extended angles. The instructional intervention was specifically designed to promote transitions among visual, symbolic, and analytical forms of reasoning through exploratory learning activities emphasizing graphical interpretation, symbolic transformation, and analytical problem solving.

Although one-group pretest–posttest designs are susceptible to internal validity threats, including maturation and testing effects, several procedures were implemented to mitigate these limitations. First, the intervention was conducted within a relatively short instructional period to minimize external influences on students' learning development. Second, identical assessment criteria and scoring rubrics were applied consistently across both testing occasions to ensure measurement stability and scoring reliability.

Research Participants

The participants consisted of 50 undergraduate students enrolled in a mathematics education program who had successfully completed an introductory trigonometry course. Participants were selected through purposive sampling based on three criteria: (a) prior completion of foundational trigonometry coursework, (b) representation across different grade point average (GPA) categories, and (c) willingness to participate in interviews and think-aloud sessions.

The inclusion of students with diverse academic backgrounds was intended to capture a broad range of cognitive characteristics and representational tendencies associated with trigonometric problem solving. Consequently, the purpose of participant selection was not statistical generalization but rather the exploration of variations in mathematical thinking patterns and representational transitions. This purposive sampling strategy facilitated a richer examination of the cognitive processes underlying students' conceptualizations of extended-angle trigonometric functions.

Instruments, Data Collection, and Procedures

Multiple sources of data were employed to obtain a comprehensive understanding of students' mathematical thinking patterns, including pretest–posttest assessments, a mathematical thinking pattern questionnaire, semi-structured interviews, and think-aloud protocols. The combination of these instruments enabled the examination of both measurable learning outcomes and the cognitive processes underlying students' reasoning.

Pretest and Posttest Assessments

The pretest and posttest were designed to evaluate students' conceptual understanding of trigonometric functions before and after the instructional intervention. Test items were intentionally constructed to be solvable through multiple representational approaches, including visualization, symbolic manipulation, and analytical reasoning. This design facilitated the identification of students' dominant thinking patterns as well as changes in representational preferences following exposure to multiple-representation learning activities as summarized in Table 1.

Table 1. Test Framework for Pre-Test and Post-Test

Aspect	General Indicator	Item No.	Sample Question
Visualization	Understanding and analyzing trigonometric function graphs	1	Determine the value of sin $225°$ using two methods: (a) unit circle, (b) related-angle identities.
Symbolic	Using identities and algebraic manipulation to solve trigonometric problems Analyzing conceptual	3	Prove that $\sin 2x = 2\sin x \cos x$ using geometric, algebraic, and unit circle approaches.
Analytical	Strong procedural accuracy and consistent interpretative reasoning	6	Solve the equation $2\, sin\ x + \sqrt{3} = 0$ in the interval $0° \leq x \leq 360°$.

The assessment framework was organized around three dimensions of mathematical thinking. The visualization dimension assessed students' abilities to interpret and analyze trigonometric graphs and geometric representations. The symbolic dimension examined students' proficiency in applying trigonometric identities and performing algebraic manipulations. The analytical dimension focused on students' capacities to establish conceptual relationships and employ logical reasoning in solving trigonometric problems. Collectively, these dimensions provided a comprehensive evaluation of students' representational competencies and conceptual development throughout the intervention.

Mathematical Thinking Pattern Questionnaire

A mathematical thinking pattern questionnaire was developed to identify students' tendencies toward visual, symbolic, and analytical approaches when engaging with trigonometric concepts. The instrument consisted of 15 Likert-scale items scored on a five-point scale ranging from 1 (strongly disagree) to 5 (strongly agree). The items assessed students' representational preferences, information-processing strategies, and approaches to mathematical problem solving.

The questionnaire framework was adapted from previous studies examining representational reasoning and cognitive flexibility in mathematics education. Content validity was evaluated by three experts in mathematics education, resulting in Aiken's (V) coefficients ranging from 0.82 to 0.91, indicating strong content validity. Internal consistency reliability analysis yielded a Cronbach's alpha coefficient of 0.87, demonstrating satisfactory reliability for research purposes presented in Table 2.

The questionnaire comprised three representational dimensions. The visualization dimension measured students' use of graphs, diagrams, unit-circle representations, and coordinate systems in understanding trigonometric concepts. The symbolic dimension assessed students' reliance on formulas, algebraic manipulation, and trigonometric identities. The analytical dimension examined students' tendencies to establish conceptual relationships, employ deductive reasoning, and prioritize conceptual understanding over rote memorization.

Table 2. Indicators and Items in the Mathematical Thinking Pattern Questionnaire

Aspect	Indicator	Item No.
Visualization	Uses graphs or diagrams to understand trigonometric concepts	1, 2
	Represents geometric visualizations when solving trigonometric problems	3, 4
	Utilizes the unit circle or Cartesian coordinates to understand angle transformations	5
	Memorizes and applies formulas in solving trigonometric problems	6, 7
Symbolic	Uses algebraic manipulation to understand trigonometric concepts	8, 9
	Prefers using trigonometric identities over visual representation	10
	Analyzes relationships between trigonometric concepts	11, 12
Analytical	Uses deductive reasoning to comprehend concepts	13, 14
	Prefers conceptual connections over rote memorization of formulas	15

Mathematical Thinking Pattern Questionnaire

Semi-structured interviews were conducted with selected participants to obtain deeper insights into the reasoning underlying their problem-solving strategies. The interviews explored how students interpreted trigonometric concepts, selected representational approaches, and navigated challenges encountered during problem solving involving extended-angle trigonometric functions.

Table 3 explains that the interview questions were organized according to the three representational dimensions investigated in this study. Questions related to visualization focused on graph interpretation, geometric reasoning, and unit-circle representations. Symbolic reasoning questions explored students' use of trigonometric identities, algebraic transformations, and formula-based approaches. Analytical reasoning questions examined students' conceptual explanations, deductive arguments, and approaches to solving trigonometric equations.

Table 3. Semi-Structured Interview Questions Categorized by Thinking Pattern

Aspect	Indicator	Item No.
Visualization	Analyzes graphs and geometric representations to understand shifts in trigonometric functions	1, 3, 5
	Uses the unit circle to determine trigonometric function values for specific angles	3, 6
Symbolic	Applies trigonometric identities to compute function values	2, 7, 9
	Writes trigonometric function equations and analyzes parameters (amplitude, period)	4, 10
Analytical	Uses deductive logic to prove trigonometric properties and identities	7, 8, 10
	Solves trigonometric equations symbolically	10

Think-Aloud Protocol

Think-aloud protocols were employed to capture students' real-time cognitive processes while solving trigonometric problems. During these sessions, participants verbalized their thoughts as they worked through mathematical tasks, allowing researchers to observe how representational transitions occurred during problem solving.

A structured rubric was developed to evaluate students' performance across visualization, symbolic, and analytical dimensions. Each dimension was assessed using a three-level scale (low, moderate, and high) presented in Table 4. The visualization category evaluated students' use of graphical representations and understanding of graph transformations. The symbolic category assessed conceptual use of formulas, trigonometric identities, and representational transitions between symbolic and visual forms. The analytical category focused on students' abilities to justify mathematical relationships, understand derivations, and flexibly coordinate multiple problem-solving strategies.

Table 4. Scoring Criteria for the Think-Aloud Protocol

Category	Specific Indicator	Score 1 (Low)	Score 2 (Moderate)	Score 3 (High)
Visualization	Uses graphs to understand trigonometric functions	Does not use graphs at all	Uses graphs only as illustrations without deep analysis	Uses graphs to comprehend concepts and recognize

Category	Specific Indicator	Score 1 (Low)	Score 2 (Moderate)	Score 3 (High)
				functional relationships
	Understands graph transformations	Fails to recognize transformations	Recognizes transformations but struggles to connect them to function concepts	Fully understands relationships between graph transformations and algebraic expressions
Symbolic	Applies trigonometric formulas in problem-solving	Uses formulas without conceptual understanding	Uses formulas with partial comprehension	Applies formulas with clear reasoning behind them
	Ability to transition to visual representation	Cannot transition to visualization	Uses visualization only when guided	Independently transitions between symbolic and visual representations to grasp concepts
Analytical	Analyzes and proves trigonometric identities before using formulas.	Memorizes identities without understanding their derivation	Partially understands derivations but does not apply them in problem-solving	Analyzes identities, understands their derivation, and applies them flexibly
	Flexibility in problem-solving strategies	Relies on a single strategy without transitioning	Attempts to transition between strategies but struggles	Seamlessly shifts between visualization, symbolic, and analytical strategies to understand concepts

Particular attention was given to students' representational flexibility, defined as their capacity to transition effectively among visual, symbolic, and analytical forms of reasoning. Higher scores reflected greater integration of multiple representations and more sophisticated conceptual understanding.

Data Analysis Techniques

Data analysis combined quantitative and qualitative techniques to generate a comprehensive account of students' mathematical thinking patterns and representational transitions.

Quantitative Analysis

Descriptive statistical analyses were initially conducted to examine the distributions of pretest and posttest scores and to identify trends in students' representational preferences before and after the intervention. These analyses provided an overview of students' conceptual understanding of extended-angle trigonometric functions and their dominant mathematical thinking tendencies.

To determine whether significant changes occurred following the intervention, inferential analyses were performed using either paired-sample (t)-tests or Wilcoxon signed-rank tests, depending on the distributional characteristics of the data (Li et al., 2021; Vierra et al., 2023). The paired-sample (t)-test was employed when assumptions of normality were satisfied, whereas the Wilcoxon signed-rank test served as a non-parametric alternative when these assumptions were violated.

To explore patterns of representational tendencies, Principal Component Analysis (PCA) was employed as a dimensionality-reduction and visualization technique rather than as a clustering procedure (Greenacre et al., 2022; Kurita, 2021). PCA was used to identify latent structures within questionnaire responses and to visualize relationships among students' representational preferences. The resulting component distributions were subsequently interpreted to identify dominant tendencies toward visualization, symbolic manipulation, or analytical reasoning.

Qualitative Analysis

Qualitative data obtained from interviews and think-aloud protocols were analyzed using thematic analysis (Wolcott & Lobczowski, 2021). The analysis involved iterative coding procedures aimed at identifying recurring themes related to students' problem-solving strategies, conceptual understanding, and representational transitions.

Particular emphasis was placed on examining how students coordinated visual, symbolic, and analytical forms of reasoning while solving trigonometric problems. Emerging themes were compared across participants to identify common cognitive patterns as well as variations in representational flexibility.

Integration of Quantitative and Qualitative Findings

Consistent with the explanatory sequential mixed-methods design, qualitative findings were used to elaborate and contextualize quantitative results. Data triangulation was employed to compare statistical patterns with evidence obtained from interviews and think-aloud protocols. This integration strengthened the credibility of the findings by ensuring that interpretations were supported by multiple sources of evidence and provided a more comprehensive understanding of students' mathematical thinking processes in learning extended-angle trigonometric functions.

Results and Discussion

This section presents the findings of the quantitative and qualitative analyses concerning students' mathematical thinking patterns in understanding trigonometric functions involving extended angles. Consistent with the explanatory sequential mixed-methods design, quantitative findings are presented first, followed by qualitative evidence derived from interviews and think-aloud protocols. The section concludes with an integrated triangulation of both datasets to provide a comprehensive account of students' representational reasoning and conceptual development.

Descriptive Analysis of Thinking Patterns and Learning Outcomes

Descriptive statistics were initially employed to examine the distribution of students' dominant thinking patterns and their performance on the pretest and posttest assessments. Based on data collected from 50 participants, changes in representational tendencies were observed following the instructional intervention.

Table 5 presents the distribution of students across the identified thinking-pattern categories before and after the intervention.

Table 5. Distribution of Students' Thinking Patterns in the Pretest and Posttest

Thinking Pattern Category	Number of Students (Pretest)	Number of Students (Posttest)
Dominant Visualization	18	14
Dominant Symbolic	16	12
Dominant Analytical	10	15
Combination (Visual-Symbolic)	4	5
Combination (Symbolic-Analytical)	2	4

As shown in Table 5, the distribution of students' thinking patterns changed following the intervention. The number of students classified within the dominant visualization category decreased from 18 to 14, while the dominant symbolic category decreased from 16 to 12 students. In contrast, the number of students exhibiting a dominant analytical orientation increased from 10 to 15. Similarly, the frequencies of students demonstrating combined representational patterns increased modestly across both visual–symbolic and symbolic–analytical categories.

These shifts suggest a movement away from reliance on a single representational approach toward greater coordination of multiple forms of reasoning. In particular, the increase in analytical and hybrid thinking patterns indicates that students became more capable of integrating visual, symbolic, and conceptual representations when solving trigonometric problems involving extended angles.

Table 6 summarizes students' performance on the pretest and posttest assessments. As shown in Table 6, students' average scores increased substantially from 54.30 on the pretest to 75.80 on the posttest. Median scores exhibited a similar increase, rising from 55 to 78. Furthermore, the reduction in standard deviation from 12.05 to 10.03 suggests a modest decrease in performance variability among students following the intervention. Collectively, these descriptive statistics indicate an overall improvement in students' conceptual understanding of trigonometric functions involving extended angles.

Table 6. Descriptive Statistics for Pretest and Posttest Scores

Statistic	Pretest	Posttest
Mean	54.30	75.80
Median	55	78
Std. Deviation	12.05	10.03
Minimum	30	50
Maximum	80	95

Analysis of Pretest–Posttest Differences

To determine whether the observed improvement was statistically significant, a paired-sample (t)-test was conducted. Prior to the analysis, normality assumptions were examined using the Kolmogorov–Smirnov test, which indicated that the score distributions did not significantly deviate from normality. Consequently, a parametric analysis was deemed appropriate.

The results, presented in Table 7, revealed a statistically significant increase in students' scores following the intervention, $(t\ (49) = -8.92, p < .001)$. To assess the practical significance of this improvement, Cohen's (d) was calculated, yielding an effect size of 1.26, which is conventionally interpreted as large. This finding indicates that the observed gains were not only statistically significant but also educationally meaningful.

Table 7. Paired-Sample (t)-Test Results

Statistical Test	t-value	p-value
Paired T-Test	-8.92	0.000**

In addition to score improvements, changes in representational tendencies were observed. The proportion of students categorized as exhibiting dominant analytical thinking increased from 20% to 30% following the intervention. Examination of students' solution processes suggested that these improvements were associated with greater coordination between graphical interpretation and symbolic reasoning, particularly when reasoning about angle transformations across quadrants. Students who demonstrated greater representational flexibility were generally more successful in explaining trigonometric relationships conceptually rather than relying exclusively on memorized procedures.

Principal Component Analysis of Representational Tendencies

To explore relationships among students' representational preferences, Principal Component Analysis (PCA) was conducted. PCA was employed as a dimensionality-reduction and visualization technique to identify underlying patterns within students' responses and to examine similarities and differences in representational tendencies.

Figure 1 displays students' distributions across the first two principal components. The horizontal axis (PC1) represents the component accounting for the largest proportion of variance in the dataset, whereas the vertical axis (PC2) represents the second-largest source of variation.

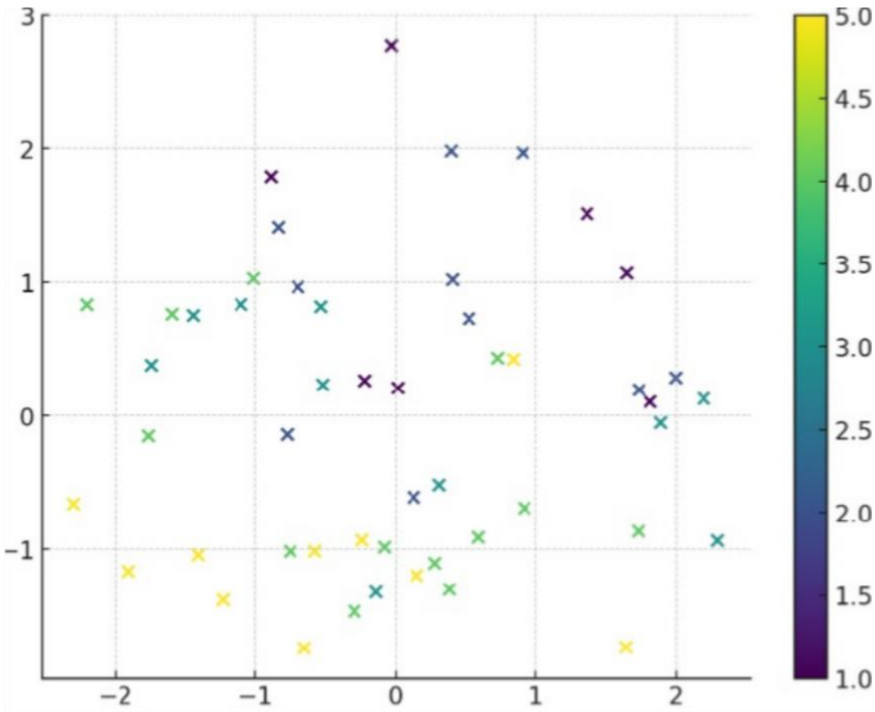

Figure 1. PCA Clustering Visualization

Students located at different positions within the PCA space exhibited distinct representational tendencies. Greater distances between points indicate stronger differences in patterns of reasoning, whereas closer points indicate greater similarity in representational preferences. The dispersion of points across the PCA plane suggests substantial heterogeneity in students' approaches to solving trigonometric problems involving extended angles.

The PCA visualization also indicates that students were not concentrated within a single region of the representational space. Instead, multiple tendencies emerged, reflecting variation in students' reliance on visualization, symbolic manipulation, and analytical reasoning. Students positioned toward the central region of the plot appeared to exhibit more balanced representational profiles, suggesting greater flexibility in coordinating multiple approaches. In contrast, students located toward the periphery tended to display stronger preferences for particular forms of reasoning.

Taken together, the PCA results provide evidence that students' mathematical thinking patterns were diverse and that representational flexibility varied considerably across individuals. These findings complement the descriptive analyses by illustrating the multidimensional nature of students' reasoning processes.

Qualitative Findings: Interviews and Think-Aloud Protocols

The qualitative analysis was conducted to explore the cognitive processes underlying students' problem-solving strategies and to provide explanatory insights into the quantitative findings. Thematic analysis of interview transcripts and think-aloud data revealed five interconnected themes presented in Table 8.

Table 8. Summary of Qualitative Analysis

Theme	Description	Example Student Quote	Interpretation
Preference for Visualization	Students understand trigonometry better by relying on graphical representations before performing numerical or symbolic calculations. They tend to draw graphs to develop intuition before using formula-based solutions.	"I prefer drawing the graph first before finding the sine or cosine value. This way, I can observe the pattern before calculating."	Students with a visual preference tend to grasp concepts through graphical representations before transitioning to symbolic methods. This indicates that they rely on spatial intuition to understand trigonometric functions.
Symbolic Tendency	Most students feel more comfortable using trigonometric identity formulas directly without first visualizing the concept. They see formulas as a quicker and more efficient way to solve problems, although in some cases, their conceptual understanding of these	"I directly use the identity formula because it's faster than drawing a graph. Once I memorize the formula, I just substitute the values."	This strategy is efficient for calculations but may hinder conceptual understanding if students rely solely on memorization without comprehending the meaning behind the formulas.

Theme	Description	Example Student Quote	Interpretation
	formulas is still limited.		
Analytical Approach	More reflective students tend to prove or derive trigonometric identities before applying them. They demonstrate higher cognitive flexibility by attempting various strategies to solve problems.	"I try to see how this identity is derived before using it to solve problems. If I understand its origin, it's easier for me to remember and apply it."	This approach indicates a deeper understanding, as students do not merely apply formulas but also strive to derive them to ensure their validity.
Difficulties in Understanding Extended Angles	Students struggle to comprehend how extended angles affect trigonometric function values.	"I get confused when I have to determine whether a negative angle is equivalent to a reflection of the graph in another quadrant."	Understanding extended angles remains a challenge, especially in linking angle transformations with graphical representations.
Changes in Thinking Strategies	Some students who initially relied on a single approach began trying alternative strategies after encountering difficulties.	"I used to always use formulas, but after seeing the graph, I better understand why the result is like that."	Changes in thinking strategies indicate students' cognitive development, especially when they begin integrating visual, symbolic, and analytical approaches.

Theme 1: Preference for Visualization

Many students reported relying on graphical representations as an entry point for understanding trigonometric relationships. Visual representations enabled them to identify patterns, anticipate function behavior, and interpret angle transformations before engaging in symbolic calculations. One student explained:

> *"I prefer drawing the graph first before finding the sine or cosine value. This way, I can observe the pattern before calculating."*

Students exhibiting this tendency frequently used graphs and unit-circle representations to construct conceptual meaning before applying formal procedures. Their reasoning was often grounded in spatial intuition and visual pattern recognition.

Theme 2: Symbolic Orientation and Procedural Efficiency

A second group of students demonstrated a strong preference for symbolic approaches. These students tended to apply formulas and trigonometric identities directly when solving problems. One participant stated:

> *"I directly use the identity formula because it's faster than drawing a graph. Once I memorize the formula, I just substitute the values."*

Although this approach often resulted in efficient computation, interview data suggested that some students relied heavily on memorized procedures without fully understanding the conceptual foundations underlying the formulas they employed.

Theme 3: Analytical Reasoning and Conceptual Justification

Students classified as analytically oriented demonstrated a greater tendency to investigate the origins and validity of mathematical relationships before applying them. For example, one student commented:

> *"I try to see how this identity is derived before using it to solve problems. If I understand its origin, it's easier for me to remember and apply it."*

Such responses indicate a deeper level of conceptual engagement, characterized by the use of justification, proof, and logical reasoning. These students frequently transitioned among visual, symbolic, and analytical representations while constructing solutions.

Theme 4: Difficulties in Understanding Extended Angles

Despite overall improvements, many students continued to experience difficulties when reasoning about extended angles and transformations across quadrants. As one participant explained:

> *"I get confused when I have to determine whether a negative angle is equivalent to a reflection of the graph in another quadrant."*

This finding suggests that connecting symbolic angle transformations with graphical interpretations remains a persistent conceptual challenge in university trigonometry learning.

Theme 5: Development of Representational Flexibility

A notable finding was the emergence of greater representational flexibility among some students following the intervention. Several participants described modifying their preferred strategies after recognizing limitations in their initial approaches. One student reflected:

> *"I used to always use formulas, but after seeing the graph, I better understand why*

the result is like that."

These accounts suggest that exposure to multiple representations encouraged students to coordinate visual, symbolic, and analytical forms of reasoning rather than relying exclusively on a single strategy.

Integration and Triangulation of Findings

The integration of quantitative and qualitative findings revealed substantial convergence across data sources. First, improvements in posttest performance were accompanied by increases in analytical and hybrid representational profiles. Students who demonstrated stronger conceptual gains were frequently those who exhibited greater flexibility in coordinating multiple forms of representation.

Second, interview data provided explanatory evidence for the observed shifts in thinking patterns. Students described moving beyond reliance on isolated visual or symbolic approaches toward more integrated forms of reasoning. This transition corresponded with the increased proportion of students categorized within analytical and combined representational groups. Third, think-aloud data confirmed that students with predominantly symbolic orientations often experienced difficulties when required to justify procedures conceptually or interpret graphical transformations without explicit formulaic support. In contrast, students displaying analytical flexibility were more likely to connect graphical, symbolic, and conceptual information during problem solving.

Overall, the triangulated findings suggest that representational flexibility plays a central role in students' conceptual understanding of trigonometric functions involving extended angles. Students who were able to coordinate visualization, symbolic manipulation, and analytical reasoning demonstrated more robust conceptual understanding and greater adaptability in problem-solving situations. These findings underscore the importance of instructional approaches that explicitly promote connections among multiple mathematical representations in university trigonometry learning.

Discussion

The findings of this study provide important insights into how university students construct conceptual understandings of trigonometric functions involving extended angles through the coordination of visualization, symbolic manipulation, and analytical reasoning. Consistent with the quantitative results, students demonstrated significant improvements in conceptual understanding following the instructional intervention, as evidenced by increases in posttest scores and shifts in representational tendencies. More importantly, the findings suggest that conceptual growth was associated not merely with procedural proficiency but with students' increasing capacity to coordinate multiple mathematical representations during problem solving.

The observed improvement in students' performance is consistent with previous studies indicating that instructional environments emphasizing representational connections can enhance mathematical understanding and facilitate more sophisticated reasoning processes (Sholahudin & Oktaviyanthi, 2023). The increase in mean posttest scores, coupled with the large effect size obtained from the pretest–posttest comparison, suggests that students developed stronger conceptual understandings of trigonometric relationships involving extended angles. Nevertheless, the findings also reveal important variations in students' learning trajectories. In particular, students who demonstrated a strong preference for symbolic reasoning exhibited comparatively slower conceptual development than students whose reasoning incorporated visual or analytical elements. This finding supports previous research suggesting that symbolic representations may impose substantial cognitive demands when students lack conceptual links to underlying graphical or geometric meanings (Begg & Pierce, 2021; Zhou & Zeng, 2022). Although many students improved procedurally, several participants continued to rely heavily on memorized formulas and algorithmic procedures. Consequently, improved performance should not automatically be interpreted as evidence of deep conceptual understanding. Rather, the findings highlight the distinction between procedural fluency and conceptual comprehension, a distinction that remains central within mathematics education research.

A particularly significant finding concerns the role of representational flexibility in students' mathematical thinking. The PCA results revealed substantial variation in students' representational tendencies and showed no rigid separation among visualization, symbolic, and analytical orientations. Instead, students' thinking patterns appeared distributed along a continuum, suggesting that mathematical reasoning is more appropriately understood as a dynamic coordination of representations rather than as a fixed cognitive style. This interpretation aligns with contemporary perspectives on mathematical cognition, which emphasize the importance of navigating among multiple representational forms when constructing mathematical meaning (Ünal et al., 2023). The absence of clearly separated representational groups further suggests that students frequently combine different forms of reasoning, even when one approach appears dominant.

The qualitative findings provide additional support for this interpretation. Students who relied primarily on visualization often used graphs and unit-circle representations as cognitive tools for constructing meaning before engaging in symbolic calculations. In contrast, symbolically oriented students tended to approach problems through formula application and algebraic manipulation, sometimes with limited attention to the conceptual foundations of the procedures employed. Analytically oriented students demonstrated a different pattern of reasoning characterized by justification, explanation, and the deliberate coordination of visual and symbolic information. These students were more likely to derive identities, examine conceptual relationships, and evaluate the validity of their solutions through multiple representational pathways.

Taken together, these findings suggest that analytical reasoning does not operate independently of visualization and symbolic manipulation; rather, it emerges through the productive integration of these representational forms. This observation extends existing research on representational fluency by demonstrating that conceptual understanding develops through coordinated transitions among representations rather than through mastery of isolated procedures or representational modes alone (Ramírez-Uclés & Ruiz-Hidalgo, 2022). The findings therefore support a view of mathematical understanding as a dynamic process of representational coordination in which students continuously move between graphical, symbolic, and conceptual perspectives while constructing meaning.

Another noteworthy finding concerns students' difficulties in reasoning about extended angles. Despite overall improvements, many participants continued to experience challenges when interpreting angle transformations across quadrants and relating symbolic angle operations to graphical representations. These difficulties were particularly evident when students were required to explain negative angles, equivalent angles, or periodic transformations. Such findings reinforce previous research indicating that students often struggle to connect formal symbolic procedures with the geometric meanings underlying trigonometric concepts (Lepore, 2024; Ramírez-Uclés & Ruiz-Hidalgo, 2022). The persistence of these difficulties suggests that conceptual understanding of extended-angle trigonometry requires more than repeated procedural practice; it requires instructional experiences that explicitly support the coordination of graphical, symbolic, and analytical reasoning.

The study also contributes theoretically by proposing representational flexibility as a central mechanism underlying conceptual understanding in trigonometry. While previous studies have often examined visualization, symbolic reasoning, or conceptual understanding as separate constructs, the present findings indicate that meaningful learning occurs through students' ability to transition among these representational domains. The proposed cognitive-transition framework conceptualizes learning as a process through which students progressively coordinate visualization, symbolic manipulation, and analytical reasoning to construct increasingly sophisticated mathematical understandings. From this perspective, representational transitions are not merely auxiliary learning strategies but constitute a fundamental component of mathematical thinking itself.

These findings have important implications for instructional practice. Mathematics instruction that privileges a single representational mode may inadvertently restrict opportunities for conceptual development. An exclusive emphasis on symbolic procedures, for example, may promote computational efficiency while limiting conceptual insight, whereas reliance solely on visual representations may not adequately support formal mathematical generalization. Consequently, instructional approaches should be designed to facilitate explicit connections among graphical interpretation, symbolic transformation, and analytical justification. Research has demonstrated that learning environments integrating multiple representations can enhance conceptual understanding and support flexible problem solving (Castro-Alonso et al., 2021; van Garderen et al., 2021; Orhani, 2024). Furthermore, technology-enhanced environments, including PhET-assisted trigonometric activities and interactive mathematical visualizations, may provide valuable opportunities for students to explore representational relationships dynamically and meaningfully (Oktaviyanthi & Sholahudin, 2023). Accordingly, university trigonometry instruction should incorporate learning sequences that systematically encourage students to translate among visual, symbolic, and analytical representations.

Several limitations should be considered when interpreting the findings. First, the use of purposive sampling and a relatively small sample size limits the transferability of the results beyond similar university mathematics education contexts. Second, the one-group pretest–posttest design does not permit strong causal inferences regarding the effectiveness of the instructional intervention because alternative explanations, including maturation, testing effects, and repeated exposure to similar problem structures, cannot be completely excluded. Therefore, the observed improvements should be interpreted cautiously as evidence of associations rather than definitive causal effects. Third, although PCA was useful for exploring representational tendencies and visualizing relationships among students, it cannot fully capture the complexity and contextual nature of students' cognitive processes. The integration of qualitative evidence partially addressed this limitation; nevertheless, future studies may benefit from employing more sophisticated longitudinal and process-oriented analytical approaches.

Future research should investigate how representational transitions develop over extended periods of learning and across different mathematical domains. Longitudinal studies could provide deeper insights into the stability and evolution of students' representational flexibility over time. Additionally, future research might examine the effectiveness of instructional interventions specifically designed to support representational transitions and explore how technological tools, collaborative learning environments, and adaptive instructional designs facilitate the development of flexible mathematical reasoning.

Finally, the findings suggest that students' conceptual understanding of extended-angle trigonometric functions is closely associated with their ability to coordinate visualization, symbolic manipulation, and analytical reasoning. By highlighting representational flexibility as a central feature of mathematical thinking, this study contributes to ongoing discussions concerning the nature of conceptual understanding in mathematics and offers a cognitive-transition framework that may inform the design of more adaptive and conceptually oriented approaches to trigonometry instruction.

Conclusion

This study investigated students' mathematical thinking patterns in understanding trigonometric functions involving extended angles through visualization, symbolic manipulation, and analytical reasoning. The findings indicate that students' conceptual understanding is characterized by dynamic transitions among multiple representational forms rather than reliance on a single mode of reasoning. Quantitative and qualitative evidence consistently demonstrated that students who exhibited greater representational flexibility were more successful in coordinating graphical, symbolic, and conceptual interpretations, leading to deeper understanding of trigonometric relationships and more adaptive problem-solving strategies.

The study contributes to mathematics education research by identifying representational flexibility as a central mechanism underlying conceptual understanding in trigonometry learning. The proposed cognitive-transition framework suggests that meaningful mathematical learning emerges through students' ability to move flexibly among visualization, symbolic manipulation, and analytical reasoning. These findings extend existing research on mathematical thinking by emphasizing representational coordination as a fundamental process in conceptual development. From a pedagogical perspective, the results highlight the importance of instructional approaches that explicitly connect graphical interpretation, symbolic transformation, and analytical justification to support students' mathematical reasoning.

Several limitations should be considered when interpreting the findings, including the use of purposive sampling, the relatively small sample size, and the absence of a comparison group. Consequently, the findings should be interpreted within the context of exploratory research in university trigonometry learning. Future studies are encouraged to employ longitudinal and comparative designs to examine the development of representational flexibility across different mathematical domains and learning environments. Such investigations may further refine the cognitive-transition framework and contribute to the design of more effective instructional practices that promote conceptual understanding and flexible mathematical thinking.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this manuscript. Furthermore, all ethical considerations, including but not limited to plagiarism, research misconduct, data fabrication or falsification, double submission or publication, and redundancy, have been fully addressed by the authors.

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