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Students' mathematical understanding in solving systems of linear equations in two variables: A Pirie–Kieren analysis

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Abstract

This study aims to analyze junior high school students' mathematical understanding across three levels of mathematical ability (high, moderate, and low) when solving systems of two-variable linear equations based on the Pirie–Kieren theory of understanding. It also examines differences in students' understanding processes to provide deeper insights into the development of mathematical understanding. The novelty of this study lies in its in-depth exploration of students' understanding processes through the Pirie–Kieren framework across different levels of mathematical ability. This study employed a qualitative descriptive design. Three eighth-grade students from SMPN 1 Pademawu were purposively selected based on mathematics achievement test results to represent high, moderate, and low mathematical ability. Data were collected through classroom observations, mathematics achievement tests, comprehension tests, and semi-structured interviews. Data were analyzed using data reduction, data display, and conclusion drawing, while methodological triangulation ensured the trustworthiness of the findings. The results revealed distinct profiles of mathematical understanding across ability levels. The student with high mathematical ability demonstrated all seven levels of understanding proposed by the Pirie–Kieren theory. The student with moderate mathematical ability demonstrated five levels: primitive knowing, image making, image having, property noticing, and formalizing. In contrast, the student with low mathematical ability demonstrated only the primitive knowing level. These findings indicate that higher mathematical ability is associated with more advanced levels of mathematical understanding, highlighting the importance of designing instruction that supports students' progression toward deeper conceptual understanding through appropriately scaffolded learning experiences.

Keywords: Mathematical understanding; Pirie-Kieren theory; Two-variable linear equation system

Introduction

Mathematical understanding is a fundamental competency in mathematics education because it enables students to interpret mathematical concepts, establish connections among ideas, and apply them effectively to problem solving. Rather than relying solely on procedural memorization, students with strong mathematical understanding can explain the underlying meaning of mathematical concepts and apply them appropriately across diverse contexts (Wijaya et al., 2018; Mulyono, 2018; Ramadhani, 2021). This competency is particularly important for interpreting multiple mathematical representations, including symbols, tables, graphs, equations, and contextual situations, and translating them into meaningful mathematical models.

One topic that demands a high level of mathematical understanding is systems of linear equations in two variables known as SPLDV. Solving SPLDV problems, particularly contextual or word problems, requires students to identify relevant information, formulate mathematical models, select appropriate solution strategies, and interpret the resulting solutions. However, previous studies have consistently reported that students experience substantial difficulties throughout this process. Puspitasari (2016), for example, found that students struggled to identify variables, translate contextual problems into mathematical equations, apply elimination and substitution methods correctly, and interpret the mathematical solutions obtained. Similar difficulties were identified at SMP Negeri 1 Pademawu, where students encountered challenges in constructing mathematical models from contextual situations and selecting appropriate procedures for solving SPLDV problems. These findings suggest that students' mathematical understanding of SPLDV remains limited and warrants further investigation.

Understanding these difficulties requires a theoretical framework capable of capturing how mathematical understanding develops rather than merely evaluating learning outcomes. The Pirie–Kieren theory provides such a perspective by conceptualizing understanding as a dynamic, recursive, and continuously evolving process. Unlike frameworks that distinguish understanding only into procedural and conceptual dimensions, the Pirie–Kieren theory describes eight interconnected layers of understanding: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventising (Laduni, 2017). Students are expected to move recursively among these layers while constructing, reorganizing, and refining their mathematical thinking. Consequently, this framework enables researchers to examine not only whether students understand mathematical concepts but also how their understanding develops throughout the problem-solving process (Lestari et al., 2021; Lanya et al., 2024). Therefore, the Pirie–Kieren framework offers a comprehensive lens for investigating students' mathematical understanding when solving SPLDV problems.

Although numerous studies have investigated students' difficulties in solving SPLDV problems, relatively few have examined these difficulties through the developmental perspective offered by the Pirie–Kieren theory while simultaneously considering differences in students' mathematical ability. Existing studies have predominantly focused on identifying procedural errors or evaluating conceptual understanding without explaining how students' progress through successive levels of mathematical understanding. Consequently, empirical

evidence describing the characteristics of students' understanding across different mathematical ability levels remains limited. Addressing this gap is essential for developing instructional strategies that support students' progression toward deeper mathematical understanding.

Accordingly, this study aims to analyze junior high school students' mathematical understanding in solving systems of linear equations in two variables through the perspective of the Pirie–Kieren theory by comparing students with high, moderate, and low mathematical ability. The findings are expected to enrich the literature on the development of mathematical understanding and provide empirical evidence to support instructional practices that foster deeper conceptual understanding in mathematics classrooms.

Methods

This study employed a descriptive qualitative research design to investigate students' mathematical understanding while solving SPLDV problems through the lens of the Pirie–Kieren theory of understanding (Corbin & Strauss, 2015). A qualitative approach was considered appropriate because it enabled an in-depth exploration of students' reasoning processes, interpretations, and the construction of mathematical understanding. The research was conducted through five stages: (1) developing the research instruments; (2) validating and revising the instruments based on expert judgment; (3) collecting data through written tests and semi-structured interviews; (4) analyzing the data; and (5) preparing the research report.

The study was conducted during the second semester of the 2023/2024 academic year at SMP Negeri 1 Pademawu. Participants were eighth-grade students from Class VIII-A who had completed instruction on Systems of Linear Equations in Two Variables, making them suitable for addressing the research objectives. A Mathematical Ability Test (MAT) was first administered to all students to classify their mathematical ability. Based on the classification proposed by Indriani and Hariastuti (2017), students scoring 80–100 were categorized as high, 60–79 as moderate, and below 60 as low mathematical ability presented in Table 1. Subsequently, three students were purposively selected to represent the three ability levels. Purposive sampling ensured that each participant adequately represented a distinct level of mathematical ability, enabling in-depth comparisons of mathematical understanding across ability groups.

Table 1. Mathematics Ability Classification

Test Score	Category
80–100	High
60–79	Medium
< 60	Low

Following participant selection, the three students completed a mathematical comprehension test consisting of SPLDV problems designed to reveal their levels of mathematical understanding. Semi-structured interviews were subsequently conducted to

clarify students' written responses, explore their reasoning processes, and obtain deeper insights into how they constructed mathematical understanding while solving the problems.

The researcher served as the primary research instrument, while the supporting instruments consisted of the MAT, the mathematical comprehension test, and a semi-structured interview protocol. Before data collection, all instruments were reviewed and validated by experts in mathematics education to ensure their content validity and alignment with the research objectives. Instrument revisions were made according to the experts' recommendations.

To enhance the credibility of the findings, time triangulation was employed. Data were collected from the same participants on different occasions using identical data collection techniques. The consistency between students' written solutions and their verbal explanations was subsequently examined to verify the stability of their mathematical understanding across time.

Data analysis was conducted using the interactive qualitative analysis model consisting of data reduction, data display, and conclusion drawing. First, MAT results were analyzed to classify students into high, moderate, and low mathematical ability groups. Second, students' written responses to the comprehension test were analyzed using indicators derived from the Pirie–Kieren theory to identify the layers of mathematical understanding demonstrated during problem solving. Third, interview transcripts were analyzed to complement and validate the interpretations obtained from the written responses.

During the data reduction stage, relevant information was selected, organized, and coded according to the eight layers of understanding in the Pirie–Kieren framework, while irrelevant information was excluded. The categorized data were then presented descriptively through narrative explanations illustrating students' understanding processes at each mathematical ability level. Finally, conclusions were drawn by synthesizing evidence from both written and interview data.

To minimize researcher bias, the interpretations were continuously compared across multiple data sources. Consistency between students' written work and interview responses was carefully examined, and the categorization of students' understanding was discussed with colleagues experienced in mathematics education research. This peer review process strengthened the credibility, dependability, and trustworthiness of the study findings.

Results and Discussion

In this study, the data analyzed consisted of mathematics ability test results, comprehension test results, and interview data. The results of the mathematics ability test administered to students in Class VIII-A of SMPN 1 Pademawu showed that, out of 25 students, one student had high mathematical ability, one student had moderate mathematical ability, and one student had low mathematical ability. For the subsequent stage of data collection, the researcher selected these three students to represent the high, moderate, and low levels of mathematical ability. Based on the mathematics ability test results and recommendations from the classroom mathematics

teacher regarding students who were able to express their ideas clearly and communicate effectively, these three students were selected as the research subjects.

Based on the subject selection criteria, the selected research subjects are presented in Table 2.

Table 2. Research Subjects

No.	Subject Code	MAT Score	Category
1.	BMT	100	High
2.	BMS	72	Medium
3.	BMR	17	Low

The selected subjects were then administered a mathematical understanding test on the topic of SPLDV. After completing the understanding test, the subjects participated in semi-structured interviews to further explore and clarify their mathematical understanding in solving SPLDV problems. The results of the analysis of students' mathematical understanding based on the comprehension test are presented as follows.

Analysis of the Mathematical Understanding of the BMT Subject in Solving Systems of Linear Equations in Two Variables on Comprehension Tests 1 and 2

From the answers and interview excerpts of the BMT subject as shown in Figure 1, it can be seen that the subject, who had high mathematical ability, was able to understand the meaning of the given problem in the first comprehension test. The subject explained that SPLDV is a system of linear equations in two variables consisting of more than one equation, each with a degree of one. In addition, the subject was able to identify and explain which equations were classified as SPLDV, and which were not. In the second comprehension test, the BMT subject followed similar steps. The subject correctly explained that SPLDV is a system of linear equations in two variables consisting of more than one equation, each with a degree of one and involving two variables. The subject was also able to identify examples that were not classified as SPLDV. This finding indicates that the subject possessed prior knowledge and understood the new definition, bringing prior knowledge to the next level of understanding through actions that assimilate or represent the definition (primitive knowing).

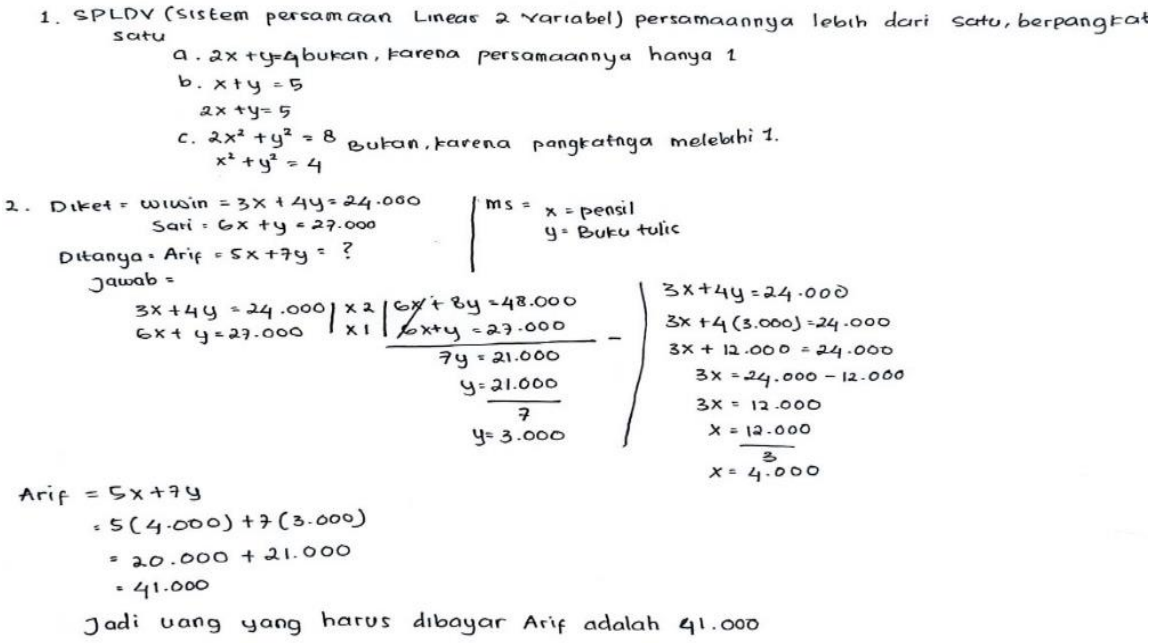


Figure 1. Comprehension Test 1 Answers of the BMT Subject

In the first comprehension test, the BMT subject was able to explain the solution steps. This was evidenced by the subject explaining that, for Problem 1, the definition was used to determine that an equation consisting of only one equation or having a degree greater than one was not classified as SPLDV. For Problem 2, the subject explained the solution by identifying the known and unknown information before applying the elimination and substitution methods as illustrated in Figure 2. In the second comprehension test, the BMT subject followed similar procedures. The subject explained that, for Problem 1, it was necessary to understand the definition first and then identify the characteristics of equations that were not classified as SPLDV. For Problem 2, the subject described the solution process in detail by identifying the known and required information before applying the appropriate formulas or concepts to solve the problem. This finding indicates that the subject was able to construct understanding from prior knowledge and apply it to new situations (image making).

In the first comprehension test, the BMT subject explained that the solution steps were used because they had been taught previously at school. When asked follow-up questions related to Problem 2, the subject was able to answer and explain that IDR 200,000 could be used to purchase 24 pencils and 12 notebooks with IDR 68,000 remaining. In the second comprehension test, the BMT subject again applied the same solution strategy because it had been learned previously and was still remembered. The subject also responded appropriately to follow-up questions by explaining that IDR 200,000 could be used to purchase 30 rulers and 10 sets with IDR 30,000 remaining, or alternatively, 35 rulers and 12 sets with no remaining balance. This finding indicates that the subject had developed internal mental images and was able to generate ideas independently without relying on concrete examples (image having).

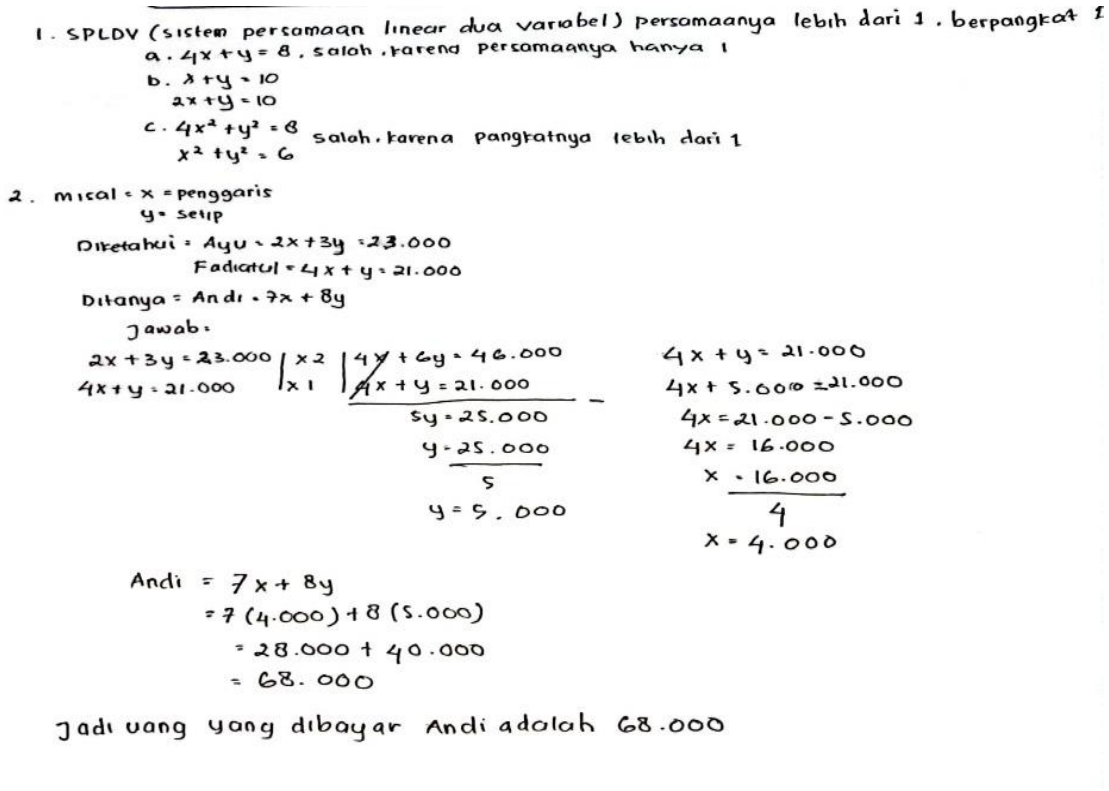


Figure 2. Comprehension Test 2 Answers of the BMT Subject

In the first comprehension test, the BMT subject identified the concepts or properties used in solving the problem, namely the elimination, substitution, and combination methods. The subject also explained the solution strategy by stating that once the values of x and y had been determined, they could be substituted to calculate the number of pencils and notebooks purchased. In the second comprehension test, the subject again identified several mathematical concepts, including elimination, substitution, subtraction, addition, multiplication, and division. The subject further explained that these solution strategies had been learned during classroom instruction and from the SPLDV textbook to determine the values of x and y. This finding indicates that the subject was able to identify and recognize the relevant properties of the problem by integrating different aspects of the mathematical concepts involved (property noticing).

In the first comprehension test, the BMT subject was able to identify and apply the mathematical concepts required to solve the problem. This was evidenced by the subject's explanation that the values of x and y were determined using the elimination and substitution methods. Similarly, in the second comprehension test, the subject explained that the elimination and substitution methods were applied to determine the values of x and y, which were then used to calculate the prices of the purchased items. This finding indicates that the subject was able to construct abstractions based on the identified mathematical properties and formulate mathematical concepts appropriately (formalising).

In the first comprehension test, the BMT subject demonstrated confidence in the obtained solution by explaining that the topic of systems of linear equations had been learned previously at school and that the answer had been verified. Likewise, in the second comprehension test,

the subject confirmed the correctness of the solution by substituting the obtained values back into the original equations. This finding indicates that the subject was able to coordinate formal operations developed at the formalising level and apply them to related mathematical situations while connecting existing mathematical understanding with new knowledge structures (observing).

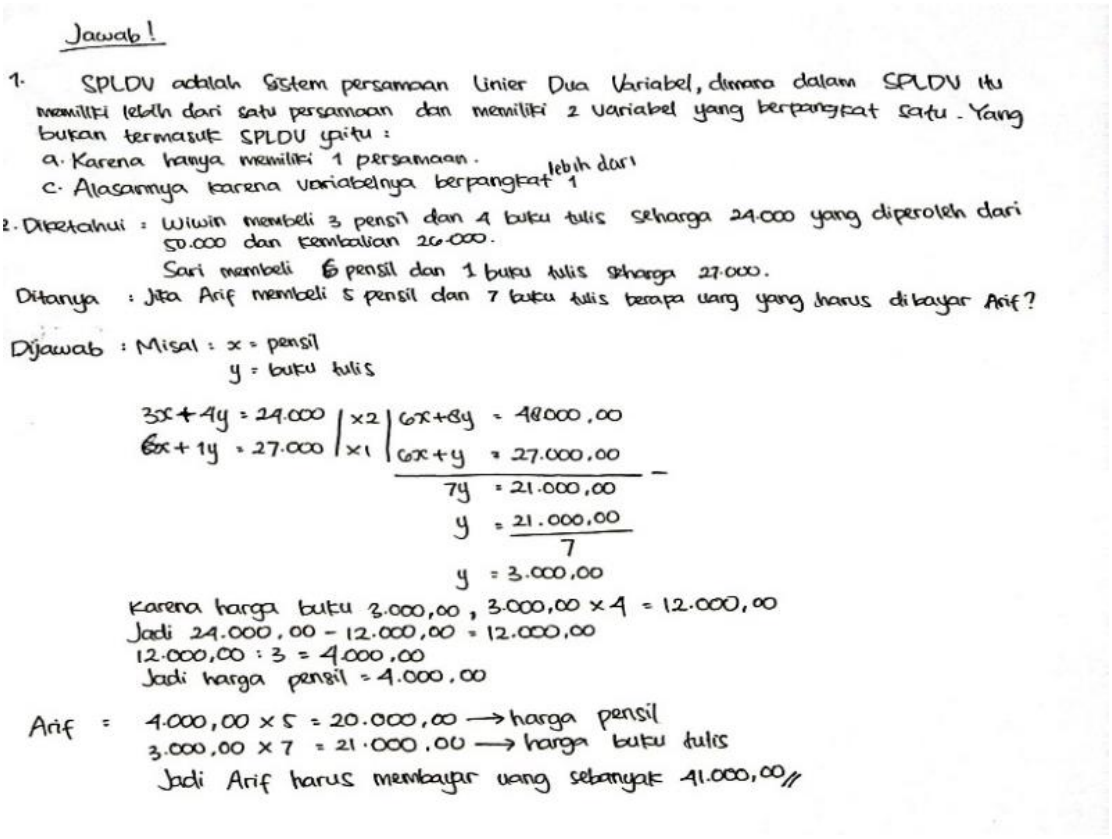
Finally, in the first comprehension test, the BMT subject verified the correctness of the obtained solution by substituting the values of x and y into the equations presented in the problem, resulting in values that were consistent with the given information. Similarly, in the second comprehension test, the subject justified the correctness of the solution by applying the same substitution process and obtaining results consistent with the values stated in the problem. This finding indicates that the subject had reached the structuring level in Pirie and Kieren (1994)'s theory, as the subject was able to connect previously constructed mathematical concepts and provide logical arguments to support the solution process.

These findings differ from those reported by Hasanah (2019), who found that students with high mathematical ability achieved all eight layers of understanding in Pirie and Kieren's theory, including the inventising level, and experienced both provocative and invocative interventions during the learning process. This difference may be attributed to several factors. First, Hasanah (2019)'s study focused on geometry concepts, which tend to encourage broader visual exploration and abstraction, thereby facilitating students' progression toward higher layers of understanding. In contrast, the present study examined students' understanding of systems of linear equations, in which students' reasoning was more procedural and verification-oriented. Second, the tasks used in Hasanah (2019)'s study provided opportunities for repeated reflection and folding-back processes, whereas the present study primarily emphasized the accuracy and justification of solutions. As explained by Pirie and Kieren (1994), the growth of mathematical understanding is recursive and highly dependent on the nature of the tasks, learning experiences, and instructional interventions. Therefore, although the BMT subject demonstrated strong logical reasoning at the structuring level, the subject did not yet exhibit the abstraction and generalization characteristics associated with the inventising level. Similar findings were also reported by Suindayati et al. (2019), who found that students' progression through the layers of understanding is strongly influenced by the complexity of mathematical tasks and opportunities for reflective thinking.

Analysis of the Mathematical Understanding of the BMS Subject in Solving Systems of Linear Equations in Two Variables on Comprehension Tests 1 and 2

From the answers and interview excerpts of the BMS subject, presented in Figure 3, it can be seen that the subject, who had moderate mathematical ability, was able to understand the meaning of the given problem in the first comprehension test. The subject understood that SPLDV is a system of linear equations in two variables consisting of more than one equation, involving two variables, and having a degree of one. In the second comprehension test, the BMS subject also demonstrated an understanding of the given problem. The subject explained

that SPLDV is a system of linear equations in two variables consisting of more than one equation, involving two first-degree variables, and correctly identified that equations a and c were not classified as SPLDV. This finding indicates that the subject possessed prior knowledge and understood the new definition, bringing prior knowledge to the next level of understanding through actions that assimilate or represent the definition (primitive knowing).


Jawab!

1. SPLDV adalah Sistem persamaan Linier Dua Variabel, dimana dalam SPLDV itu memiliki lebih dari satu persamaan dan memiliki 2 variabel yang berpangkat satu. Yang bukan termasuk SPLDV yaitu :
a. Karena hanya memiliki 1 persamaan.
c. Alasannya karena variabelnya berpangkat lebih dari 1

2. Diketahui : Wiwin membeli 3 pensil dan 4 buku tulis seharga 24.000 yang diperoleh dari 50.000 dan kembalian 26.000.
Sari membeli 6 pensil dan 1 buku tulis seharga 27.000.
Ditanya : Jika Arif membeli 5 pensil dan 7 buku tulis berapa uang yang harus dibayar Arif?
Dijawab : Misal : x = pensil
y = buku tulis

$3x+4y = 24.000 \mid \times 2 \mid 6x+8y = 48000,00$
$6x+1y = 27.000 \mid \times 1 \mid 6x+y = 27.000,00$
$7y = 21.000,00$
$y = \frac{21.000,00}{7}$
$y = 3.000,00$

Karena harga buku 3.000,00, 3.000,00 × 4 = 12.000,00
Jadi 24.000,00 − 12.000,00 = 12.000,00
12.000,00 : 3 = 4.000,00
Jadi harga pensil = 4.000,00

Arif = 4.000,00 × 5 = 20.000,00 → harga pensil
3.000,00 × 7 = 21.000,00 → harga buku tulis
Jadi Arif harus membayar uang sebanyak 41.000,00//

Figure 3. Comprehension Test 1 Answers of the BMS Subject

In the first comprehension test, the BMS subject was able to explain the steps used to solve the problem. This was evidenced by the subject explaining that, for Problem 1, the definition was used to determine which equations were not classified as SPLDV, whereas for Problem 2, the solution began by identifying the known and required information before applying the appropriate method as shown in Figure 4. In the second comprehension test, the BMS subject followed a similar procedure. The subject explained that, for Problem 1, the definition was first considered before selecting the equations that were not classified as SPLDV. For Problem 2, the subject identified the known and required information in the word problem before applying the previously learned method to obtain the required solution. This finding indicates that the subject was able to construct understanding from prior knowledge and apply it to new situations (image making).

In the first comprehension test, the BMS subject explained that the solution steps were applied because they had previously been taught at school. When asked follow-up questions related to Problem 2, the subject was able to answer and explain that IDR 200,000 could be used to purchase 10 pencils and 14 notebooks with IDR 18,000 remaining. In the second

comprehension test, the BMS subject again explained that the same solution strategy was used because it had been learned previously. When responding to follow-up questions related to Problem 2, the subject explained that IDR 200,000 could be used to purchase 14 rulers and 18 sets of chips with IDR 54,000 remaining. This finding indicates that the subject had developed internal mental images and was able to generate ideas independently without relying on concrete examples (image having).

In the first comprehension test, the BMS subject identified the concepts or properties used in solving the problem, namely the elimination and substitution methods. However, the subject was unable to explain the underlying reasoning logically when describing how the solution idea was developed. Nevertheless, the subject was able to combine different aspects of the mathematical concepts to identify the relevant characteristics of the problem. Therefore, the BMS subject fulfilled the property noticing indicator.

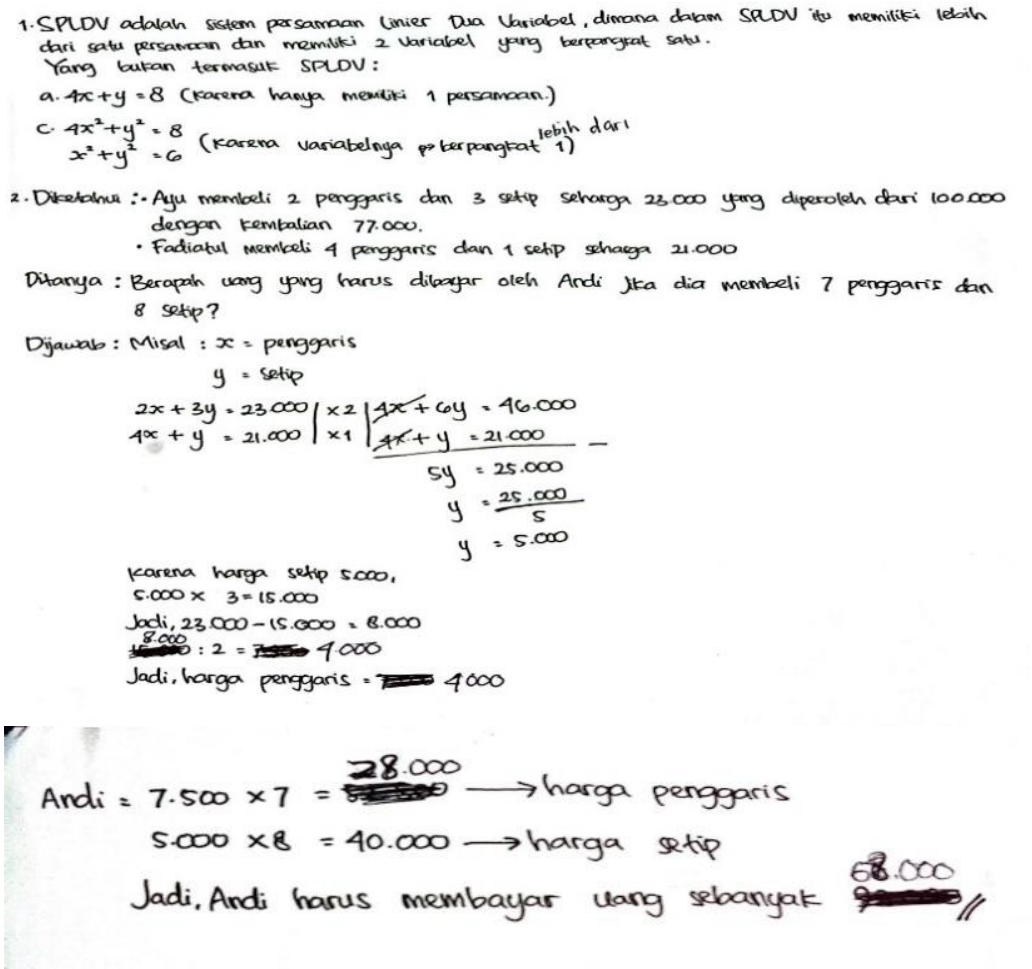


Figure 4. Comprehension Test 2 Answers of the BMS Subject

Finally, in the first comprehension test, the BMS subject was able to identify and apply the mathematical concepts required to solve the given problem because the subject recalled and understood the material that had previously been learned in class. The subject explained that

the elimination and substitution methods were used to determine the values of x and y. Likewise, in the second comprehension test, the subject demonstrated an understanding of the procedures and mathematical concepts used to solve the problem, as evidenced by the explanation regarding the application of the elimination and substitution methods. This finding indicates that the subject had reached the formalising level in Pirie and Kieren (1994)'s theory because the subject was able to formulate mathematical ideas and construct abstractions from previously understood concepts. At this stage, students are not only able to perform procedures but are also capable of explaining the relationships among concepts and the reasoning underlying the solution process.

The findings of this study differ from those reported by Hasanah (2019), who found that subjects with a moderate understanding of geometry concepts were only able to reach the structuring level and experienced only provocative intervention. This difference may be influenced by the characteristics of the mathematical content and the nature of students' prior experiences. Hasanah (2019)'s study focused on geometry concepts, which require high levels of visualization, spatial reasoning, and abstraction. Consequently, students with moderate mathematical ability often experienced difficulties in constructing abstract geometric relationships, limiting their progression to higher layers of understanding. In contrast, the present study examined systems of linear equations, in which students relied on procedural algebraic reasoning that had been repeatedly practiced during classroom instruction. Such experiences supported students in constructing formal mathematical representations more systematically and enabled them to achieve the formalising level.

This interpretation is supported by Pirie and Kieren (1994), who argued that mathematical understanding develops recursively through interactions among prior knowledge, learning experiences, and reflective activity. Ayu (2021) also explained that students' understanding develops dynamically through folding-back processes, in which previously constructed knowledge is revisited to support the development of more formal mathematical ideas. Furthermore, Indriani and Hariastuti (2017) found that students' understanding in the Pirie–Kieren model frequently develops within the image making and formalising layers when students are able to identify patterns and construct rules from previously learned concepts. Similarly, Alaiya et al. (2024) reported that students' progression through the layers of mathematical understanding is strongly influenced by scaffolding, problem structure, and opportunities to connect procedural and conceptual knowledge. In addition, Isna et al. (2018) emphasized that different mathematical topics contribute differently to students' development of mathematical understanding because geometry learning often requires more complex representational transformations than algebraic problem solving. Therefore, although the BMS subject was categorized as having moderate mathematical understanding, the subject was still able to attain the formalising level because the algebraic structure of the task facilitated the abstraction and formulation of mathematical ideas.

Analysis of Mathematical Understanding of BMR Subjects in Solving Two-Variable Linear Equation System Problems on Comprehension Test 1 and Comprehension Test 2

From the answers and interview excerpts of the BMR subject as shown in Figure 5, it can be seen that the subject, who had low mathematical ability, was able to understand the meaning of the given problem in the first comprehension test. The subject was able to explain the definition of SPLDV and identify examples that were not classified as SPLDV. This was evidenced by the subject explaining that SPLDV is a system of linear equations in two variables consisting of more than one equation, involving two variables, and having a degree of one, while equations a and c were not classified as SPLDV. In the second comprehension test, the BMR subject also demonstrated an understanding of the given problem. The subject explained the meaning of SPLDV and identified the equations that were not classified as SPLDV, stating that SPLDV is a system of linear equations in two variables consisting of more than one equation, involving two variables, and having a degree of one, whereas equations a and c were not included. This finding indicates that the subject possessed prior knowledge and understood the new definition, bringing prior knowledge to the next level of understanding through actions that assimilate or represent the definition (primitive knowing). This finding is consistent with Hasanah (2019), who reported that students with a low level of understanding of geometry concepts reached only the first level of Pirie and Kieren's theory, namely primitive knowing, and experienced only one form of intervention, namely invocative.

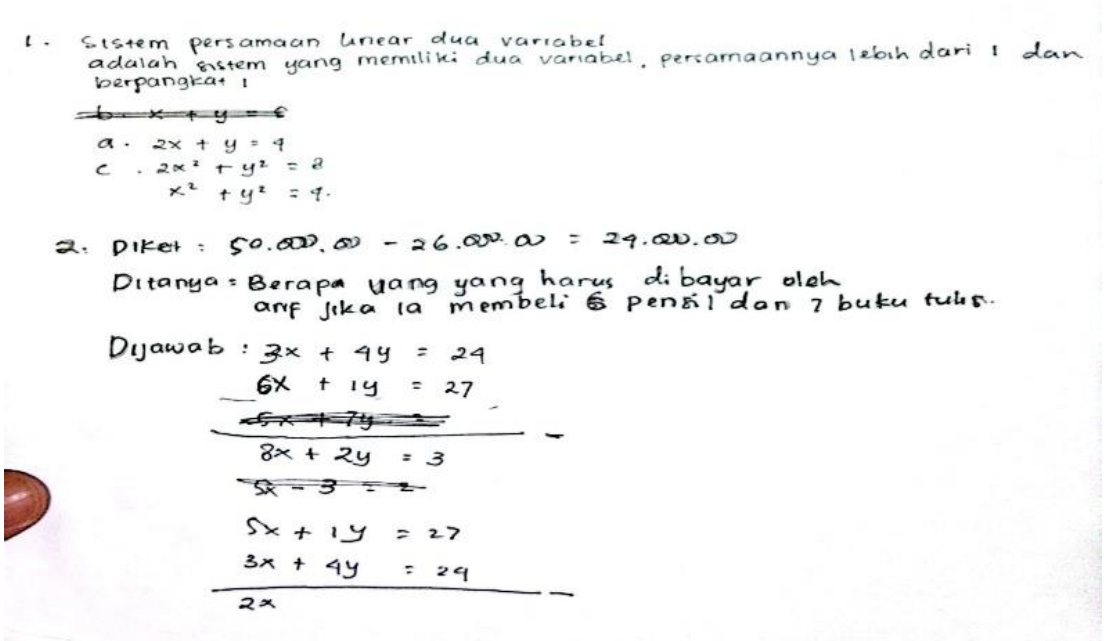


Figure 5. Comprehension Test 1 Answers of the BMR Subject

Based on the findings of this study, differences were identified in the layers of understanding described in Pirie and Kieren's theory when solving SPLDV problems. The findings obtained from the analysis of the BMT subject's mathematical understanding in

solving SPLDV on Comprehension Tests 1 (Figure 5) and 2 (Figure 6) differ from those reported by Hasanah (2019), who found that subjects with high mathematical ability achieved all eight levels of understanding in Pirie and Kieren's theory and experienced both provocative and invocative interventions. Meanwhile, the findings related to the BMS subject differ from Hasanah's (2019) study, which reported that subjects with a moderate understanding of geometry concepts reached the structuring level and experienced only provocative intervention. In contrast, the findings for the BMR subject are consistent with Hasanah's (2019) study, which found that subjects with a low level of understanding of geometry concepts reached only the first level, namely primitive knowing, and experienced only one form of intervention, namely invocative.

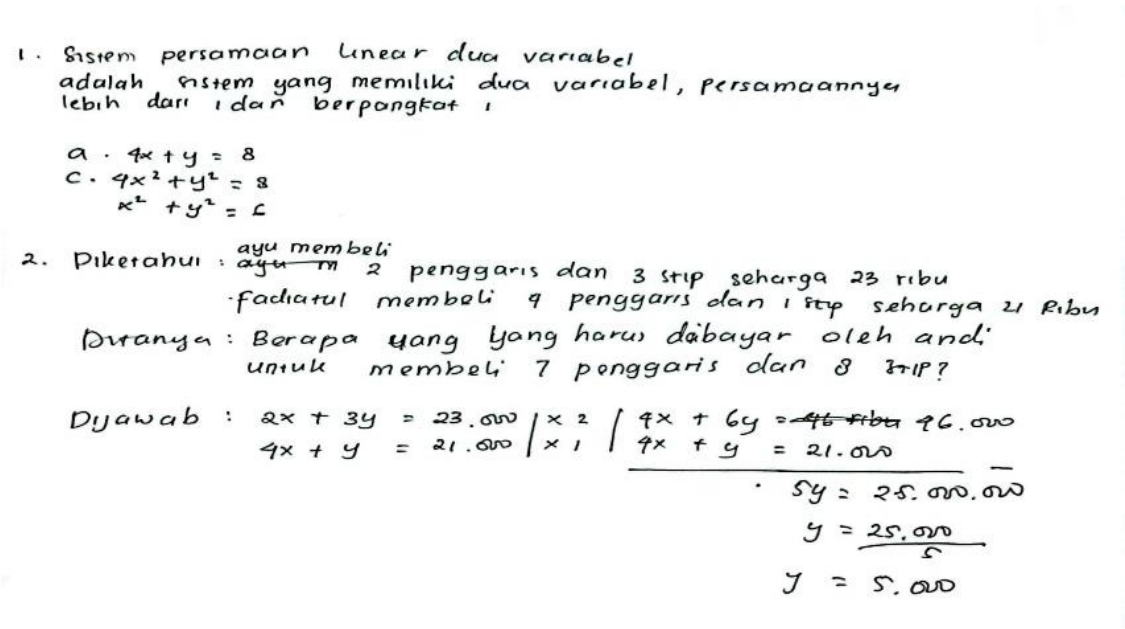


Figure 6. Comprehension Test 2 Answers of the BMR Subject

Previous studies conducted by Sidik and Sudiana (2023) revealed that students' levels of mathematical understanding differ according to their mathematical ability, namely high, moderate, and low levels. The findings of the present study support this result because differences were also identified in the layers of mathematical understanding achieved by the subjects when solving SPLDV problems. Students with higher mathematical ability tended to reach higher layers of understanding, such as structuring and formalising, whereas students with lower mathematical ability experienced difficulties in connecting concepts, constructing mathematical models, and verifying the correctness of their solutions. These findings indicate that the development of students' mathematical understanding in Pirie and Kieren's theory is strongly influenced by prior conceptual mastery and students' ability to reflect on their own thinking processes.

The present study also found that several students experienced difficulties in transforming contextual problems into mathematical representations and drawing appropriate conclusions from the obtained solutions. This finding is consistent with the studies conducted by Putri and Priatna (2023) and Yunita et al. (2024), which reported that students frequently struggle to

translate contextual situations into algebraic forms because they do not fully understand the relationships among the mathematical concepts involved in the problem. In the present study, these difficulties were particularly evident among subjects with moderate and low mathematical ability, who tended to focus primarily on procedural calculations without fully understanding the meaning of the mathematical representations they constructed. As a result, students were often unable to explain the reasoning underlying their solutions or evaluate the correctness of their answers.

Furthermore, the findings suggest that students' mathematical habits of mind also influenced the development of their mathematical understanding. Several subjects did not carefully review their answers and demonstrated limited reflective thinking during the problem-solving process. This finding supports Pebriana and Imami (2023), who stated that weak mathematical habits of mind can inhibit students' conceptual understanding, particularly in tasks requiring analysis, reasoning, and verification. Similarly, Safitri (2017) explained that students who are not accustomed to reflective problem solving tend to make procedural errors and fail to evaluate the correctness of their solutions. Within the context of Pirie and Kieren's theory, reflective thinking and folding-back processes are essential because students need to revisit their prior understanding to construct more advanced mathematical ideas (Pirie & Kieren, 1994; Moleong, 2010). Therefore, students who rarely engage in reflective activities may experience difficulties progressing toward higher layers of understanding.

Based on these findings, the recommendations proposed in this study are directly derived from the difficulties experienced by the subjects during the problem-solving process. Since students experienced difficulties connecting prerequisite concepts with new mathematical ideas, teachers need to reactivate students' prior knowledge before introducing more complex concepts. This is important because the growth of mathematical understanding develops recursively through interactions between prior knowledge and new learning experiences (Pirie & Kieren, 1994). In addition, because several subjects experienced difficulties explaining their reasoning and constructing formal mathematical representations, teachers are encouraged to provide scaffolding strategies, such as guiding questions, reflective prompts, worked examples, and constructive feedback. Such support can facilitate students' folding-back processes and help them construct deeper conceptual understanding (Moleong, 2010).

Moreover, the findings of this study indicate that mathematics instruction should not merely emphasize the procedural completion of problems but should also encourage students to explain their reasoning, verify their solutions, and connect mathematical ideas conceptually. Conceptually oriented instruction and reflective mathematical activities are important because they can help students develop deeper mathematical understanding and improve their problem-solving abilities (Asih et al., 2021). Similarly, Dalal and Rani (2013) emphasized that learning environments focusing on conceptual understanding can improve students' mathematical reasoning and support the development of meaningful understanding. Therefore, teachers need to design learning activities that encourage reflection, conceptual connections, and active mathematical reasoning so that students can achieve higher layers of understanding in Pirie and Kieren's theory.

Conclusion

Students' mathematical understanding in solving SPLDV problems varies according to their levels of mathematical ability and reflects the hierarchical nature of understanding proposed in Pirie and Kieren's theory. The findings indicate that higher mathematical ability enables students to progress through more layers of understanding, whereas lower mathematical ability limits the development of deeper conceptual understanding and mathematical reasoning. These results demonstrate that the growth of mathematical understanding is closely related to students' prior knowledge, conceptual mastery, and capacity to construct and reflect on mathematical ideas.

Students with high mathematical ability demonstrated understanding across almost all layers of the Pirie and Kieren theory, ranging from primitive knowing to observing and structuring, although they had not yet reached the inventising layer. These students were able to understand mathematical concepts both procedurally and conceptually, justify their solutions, connect mathematical ideas, and reflect on their reasoning processes. Their inability to attain the inventising layer suggests that generating new mathematical structures or developing original solution strategies remains a cognitively demanding process, even for students with high mathematical ability. In contrast, students with moderate mathematical ability demonstrated understanding up to the formalising layer. They were generally able to explain procedures, apply mathematical concepts, and solve related problems; however, their understanding remained limited in terms of deeper abstraction, logical justification, and reflective reasoning. This finding indicates that students with moderate mathematical ability tend to rely more on procedural understanding than on constructing fully interconnected conceptual structures and therefore require instructional support that promotes reflective thinking and explicit mathematical reasoning.

Students with low mathematical ability demonstrated understanding only at the primitive knowing layer. Although they were able to identify basic definitions and simple examples related to SPLDV, they experienced difficulties in representing mathematical ideas, connecting concepts, and applying mathematical properties to problem solving. These findings suggest that limited prior knowledge and weak conceptual foundations hinder students from progressing to higher layers of understanding. Consequently, students with low mathematical ability require more intensive scaffolding, guided practice, and contextual learning experiences to strengthen their conceptual understanding and support their mathematical development.

Overall, this study confirms that Pirie and Kieren's theory provides a valuable framework for analyzing students' mathematical understanding across different levels of mathematical ability. The findings imply that mathematics instruction should not focus solely on obtaining correct answers but should also foster conceptual understanding, mathematical reasoning, reflection, and meaningful connections among mathematical ideas. Accordingly, teachers are encouraged to design learning activities that are aligned with students' existing levels of understanding and that provide appropriate opportunities for reflection, conceptual exploration, and progressive development toward higher layers of mathematical understanding.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. All authors have contributed significantly to the research and the preparation of this manuscript, and they have approved the final version submitted for publication. The authors confirm that the manuscript is original, has not been published previously, and is not currently under consideration by any other journal. Furthermore, all ethical standards have been strictly observed throughout the research process. The authors affirm that there are no issues of plagiarism, data fabrication, falsification, redundant publication, or other forms of research misconduct. All sources used in this study have been properly acknowledged and cited.

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