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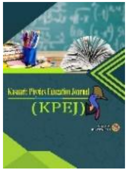

Electric Field Inside of a Homogeneous Charged Spherical Surface

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Abstract: *It is widely known that the electric field inside an uniformly charged close surface is null at any point inside the body with independence of its geometry. This result, which is considered as a Physics theorem, is typically justified by applying symmetry arguments based on Gauss law inside the surface. However, a formal proof is not frequently given due to its mathematical complexity. In the present article a formal study of the electric field inside a uniformly charged spherical surface is given. The solution is presented through three different approaches: derivation of the electric potential inside the body, superposition of the electric field contributions from infinitesimal ring-elements, and finally, direct integration of the electric field Coulomb's law for all the surface elements of the sphere. In all cases, the same result is achieved, confirming a null electric field inside the sphere. This formal solution that is not given in the most classical books of General Physics and Electromagnetism, might be interesting in an academic context, for undergraduate students and professors of science and engineering curricula.*

Keywords: *Coulomb's law, electric potential, electrostatics, general physics, superposition principle*

Medan Listrik di Dalam Permukaan Bola Bermuatan Homogen

Abstrak: Telah diketahui secara luas bahwa medan listrik di dalam permukaan tertutup bermuatan seragam adalah nol di titik mana pun di dalam benda tersebut, terlepas dari geometrinya. Hasil ini, yang dianggap sebagai teorema Fisika, biasanya dibenarkan dengan menerapkan argumen simetri berdasarkan hukum Gauss di dalam permukaan. Namun, pembuktian formal tidak sering diberikan karena kompleksitas matematisnya. Dalam artikel ini, diberikan studi formal medan listrik di dalam permukaan bola bermuatan seragam. Solusinya disajikan melalui tiga pendekatan berbeda: derivasi potensial listrik di dalam benda, superposisi kontribusi medan listrik dari elemen-elemen cincin yang sangat kecil, dan terakhir, integrasi langsung hukum Coulomb medan listrik untuk semua elemen permukaan bola. Dalam semua kasus, hasil yang sama tercapai, yang mengonfirmasi medan listrik nol di dalam bola. Solusi formal ini yang tidak diberikan dalam buku-buku Fisika Umum dan Elektromagnetisme yang paling klasik, mungkin menarik dalam konteks akademis, bagi mahasiswa dan dosen kurikulum sains dan teknik.

Kata Kunci: elektrostatika, fisika umum, hukum Coulomb, prinsip superposisi, potensial listrik

INTRODUCTION

The study of the electric field created by charged extended bodies is a classical problem that address the most books of General Physics and Electromagnetism (Feynman et al., 1972; Eisberg & Lerner, 1988; López, 2002; Tipler & Mosca, 2010; del Vigo & Villarino, 2020). Symmetry of the body, charge uniformity and the place where electric field is studied, determines the difficulty of the problem's solution (Griffiths, 2017; Marón & del Vigo, 2024). A uniformly charged closed surface (S), with constant density σ and enclosing volume V (Figure 1) is electrically equivalent to a conducting body, of the same geometry and total charge (Q), in electrostatic equilibrium. Indeed, a conducting body, by its own nature, permits the free movement of charge. Then, the excess charge, whether carried by electrons or holes, moves free through the conductor due to its electrostatic repulsion, eventually accumulating on the body's surface. At the end of this process, electrostatic equilibrium is reached when the charges redistribute uniformly in such a way that the distance between them is maximized, what corresponds to a uniformly distribution along the body's surface.

The electric field inside a conductor in electrostatic equilibrium (or inside a closed surface uniformly charged with the same charge and geometry) is zero. This theorem that is experimentally known for over 200 years is commonly justified in most General Physics textbooks using Gauss's law and symmetry arguments. Indeed, consider a conducting body with charge Q and frontier S in electrostatic equilibrium, and then, with a constant charge density σ as illustrated at Figure 1. Additionally, consider a Gaussian surface (S') inner to the conducting body with the same shape (see Figure 1). Notice that, both surfaces are nearly coincident. Since all the electric charge resides on the surface S, then, there is no charge within S'. Therefore, by applying Gauss's law yields:

$$\Phi = \int_{S'} \vec{E} \cdot d\vec{S} = \frac{q_{in}}{\varepsilon_0} = 0 \tag{1}$$

Where $\vec{E}$ represents the electric field crossing S', and $q_{in} = 0$ is the (null) electric charge enclosed by S'. The most General Physics textbooks, such as, Burbano de Ercilla et al. (2003), Tipler and Mosca (2010) or Serway & Vuille (2018), at this point, justify a null electric field inside the conductor by this result. According to these references, this result justify the null electric field contribution inside the body since the electric field flux (Φ) is null over any non-zero surface (S') inside the charged body. However, it might be notice that this argument is not conclusive as there exists non-zero vector functions whose definite integral over a close surface yields zero; for instance, an odd-type function with symmetric integration limits.

Strictly speaking, studies in the Solid State Physics field have concluded that the charge on a conductor in electrostatic equilibrium is confined to a region extending up to a maximum of two atomic diameters inside the body surface (Feynman et al., 1972). This should be the geometrical limit to be taken for the Gaussian surface (S').

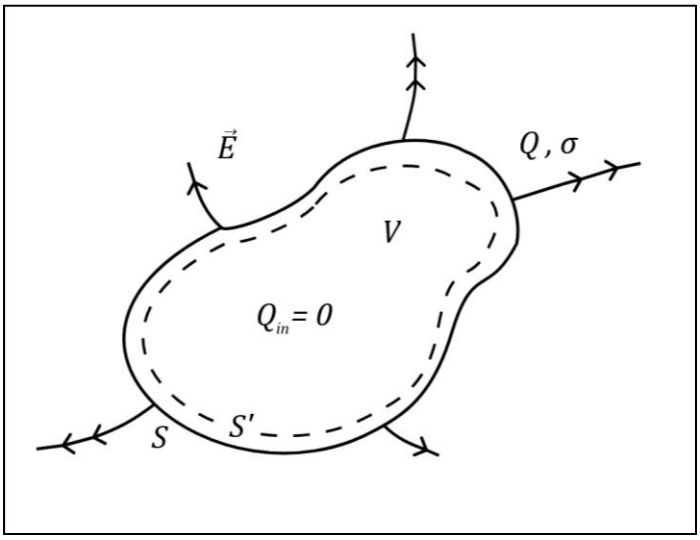

Figure 1. Charged Conducting Body in Electrostatic Equilibrium. Body Volume (V) Inside its Boundary (S) Which is Uniformly Charged with a Constant Density (σ). The Gaussian Surface (S') is Inner and Almost Coincidence With S

A more advanced Physics textbooks (Landau & Lifshitz, 1981; Purcell, 1988; Quesada et al., 2000; Griffiths, 2017) establish that, the lack of free charge movement (electrons or holes) inside a conductor in electrostatic equilibrium justify by itself, the absence of electric field inside the body. Indeed, if $\vec{E} = 0$ within the conductor, and in the absence of other external forces, the charges may only move with a uniform and straight movement; however, given the finite size of the body, the charges must remain at rest under this condition. Nevertheless, it should be notice that, this argument is constrained by the conductive properties of the material, since residual electric fields within the body may be insufficient to displace any charge depending on the material's conductivity. Therefore, it could exist some weak electric field within the body (e.g., at the local scale and near atoms), incapable to mobilize charge. In this context, Feynman et al. (1972) provides an interesting discussion about several experimental works (dating back to Benjamin Franklin) performed with the aim of measure any electric field intensity inside a charged conducting sphere in electrostatic equilibrium. All these experimental works have provided upper bounds for the magnitude of any possible electric field inside the conductor. In this framework, assuming a null electric field intensity inside a uniformly charge spherical surface is a necessary to justify the field as a central force that follows a $1/r^2$ dependence. The existence of a non-zero electric field inside the sphere would imply (Feynman et al., 1972) a deviation from the inverse-square Coulomb's law, arising an exponent that would differ from 2.

A more mathematically advance justification for the absence of an electric field within a conductor in electrostatic equilibrium involves the application of the uniqueness theorem for Laplace's equation over the region that defines its volume V and frontier S (Figure 1). This method establishes that the electric field inside the conductor must vanish, even when the system is subject to external electric fields arising from distant sources. This result is particularly interesting in engineering since it provides the theoretical basis of electrostatic shielding applied by a conducting body against external fields. A detailed derivation of this result can be found in Purcell (1988) or Griffiths (2017). However, this mathematical proof rests on the premise that the charge density spreads over the surface according to the same criteria previously discussed, and then, similar objections can be argued.

As a consequence, the possibility of electric field existence inside a conductor in electrostatic equilibrium should be still considered as a potentially open question, despite the strong experimental evidence and robust mathematical justifications presented in the literature regarding on General Physics and Electromagnetism. In the particular case of a uniformly charged spherical surface (Figure 2) of radius R, and constant charge density σ, most of the General Physics textbooks present a solution based on Gauss's Law, equation (1), assuming that the electric field within the sphere must to be radial, according to its symmetry, in a spherical coordinates system. Then, for any Gaussian concentric spherical surface (S') of radius $r < R$ (Figure 2), the following result arises (2).

$$\Phi = \int_{S'} \vec{E} \cdot d\vec{S} = \int_{S'} E\hat{r} \cdot dS\hat{r} = E \int_{S'} dS = E4\pi r^2 = \frac{q_{in}}{\varepsilon_0} = 0 \Leftrightarrow E = 0 \qquad (2)$$

Being $\hat{r}$ the (radial) unitary vector parallel to the electric field and gaussian surface at any point inside the sphere. Since $\hat{r} \cdot \hat{r} = 1$, and given that, the electric field intensity remains constant over any concentric Gaussian surface this result is justified. Remarks that, any point of any concentric Gaussian surface remains to a constant distance of the spherical charge distribution (Figure 2).

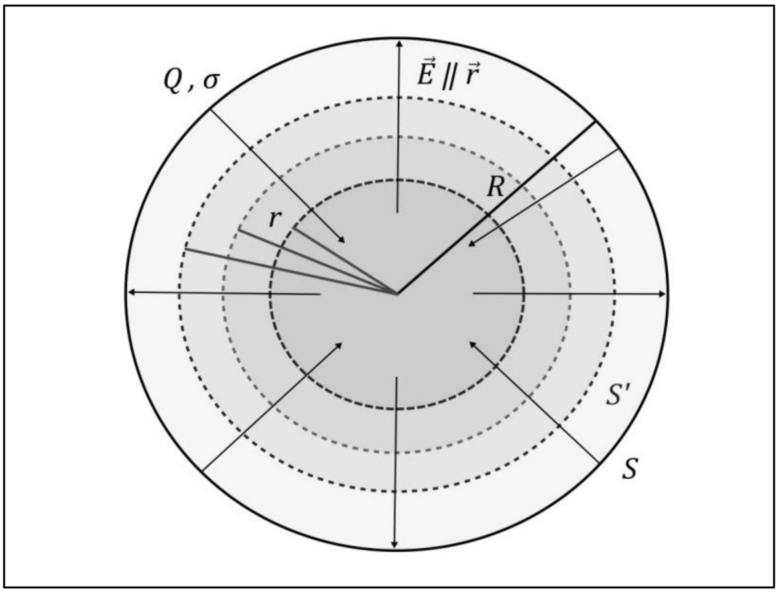

Figure 2. Representation of the Radial Electric Field Inside a Spherical Uniformly Charge Surface of Radius R and Constant Density σ. Several Gaussian Concentric Surfaces (S') of Radius $r < R$ are Depicted

Nevertheless, an often unnoticed consideration in most textbooks is the theoretical possibility that the electric flux could vanish not because the field itself is zero, but also if the radial field components are symmetrically opposed as represented at Figure 2. This is a possibility if the half of the field is directed inward and the other half outward the Gaussians spheres. In such a case, the net electric flux through the Gaussian surface would cancel out, giving arise Equation (2) to zero, since (3).

$$\Phi = \int_{S'} \vec{E} \cdot d\vec{S} = \int_{S'/2} E\hat{r} \cdot dS\hat{r} + \int_{S'/2} -E\hat{r} \cdot dS\hat{r} = 0 \qquad (3)$$

Which purely vanishes due to the field configuration symmetry. In this case, it is not necessary a zero electric field intensity solution. Even though this possibility is apparently forced or artificial, it is normally omitted from the standard treatments found in the consulted literature. Nonetheless, it leaves open the theoretical chance for non-zero electric field within the sphere interior.

Inside this context, this work provides a rigorous mathematical proof of the null electric field intensity inside a uniformly charged spherical surface. The high degree of symmetry inherent to the sphere permits a direct analytical treatment of the fundamental electrostatics equations leading to this result. The solution is derived independently through three distinct approaches which are based on the superposition principle: (i) solving the electrostatic potential throughout the interior volume of the sphere, (ii) integrating the electric field contributions from all infinitesimal ring-elements that compose the entire sphere, and (iii) performing a direct integration of Coulomb's law over all infinitesimal surface elements. In all three cases, the analysis unambiguously confirms that the electric field vanishes at every point within the sphere.

The authors think that this work is interesting in an academic context for undergraduate students and professors who wish deepen to this formal solution that is not frequently given in classical books of Physics and Electromagnetism.

MATERIALS AND METHODS

Basis of the Cartesian coordinates system is used $(\hat{\imath}, \hat{\jmath}, \hat{k})$ along the equations involved in this article. This is a necessary condition to integrate (easily) the equations since these unitary vectors are constant in the Euclidean space, and them, they can be extracted from the integrals. However, due to the system symmetry the Cartesian components are transformed in spherical coordinates, such as (4).

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases} \tag{4}$$

Where $r \in \mathbb{R}^+$ represents the distance to the origin, $\theta \in [0, \pi]$ is the colatitude angle and $\phi \in [0, 2\pi]$ is the longitude angle (sometime called azimuth). The sphere center is assumed to be the origin of the reference system in any case.

The spherical surface of radius R is assumed to be positive and uniformly charge with a constant density σ. The electric field is calculated inside the surface for any $r < R$. The solution of the problem is given by three different paths of increasing mathematical difficulty. Firstly, by determining the electric potential function that is a constant magnitude inside the sphere. Since the electric potential is a scalar function, a unique integral is involved in this case. In a second place, electric field inside the sphere is obtained as the superposition of the finite ring-elements that shape the entire sphere along its symmetry axis. This method involves a vector function integration by a unique direction. Finally, a direct integration of the electric field (with its three components) is performed for all the finite surface elements of the sphere. This procedure is probably the most rigorous since it involves the direct integration of the Coulomb's law for all the points inside the sphere. All these techniques involve the application of the superposition principle. Permittivity of the free space (ϵ_0) is used instead of Coulomb's constant along the equations presented in this article (5).

$$k = \frac{1}{4\pi\epsilon_0} \tag{5}$$

All the calculation presented in this article were manually performed. Classical integration techniques over multi-variable functions were applied (Pauly, 1909; Marsden & Tromba, 1991; Spivak, 1996). Additionally, some of the integrals involved were checked with symbolic package of MATLAB software in a personal computer: CPU Intel Core i7-4700MQ with 2.40GHz and 20GB RAM memory. English language writing was improved in some parts of the article using AI.

RESULTS AND DISCUSSION

The Electric Potential Method

Consider a spherical surface with radius R and uniformly distributed total charge Q with a constant density σ. The sphere center is assumed to be the origin of the reference system. Figure 3 represents this system.

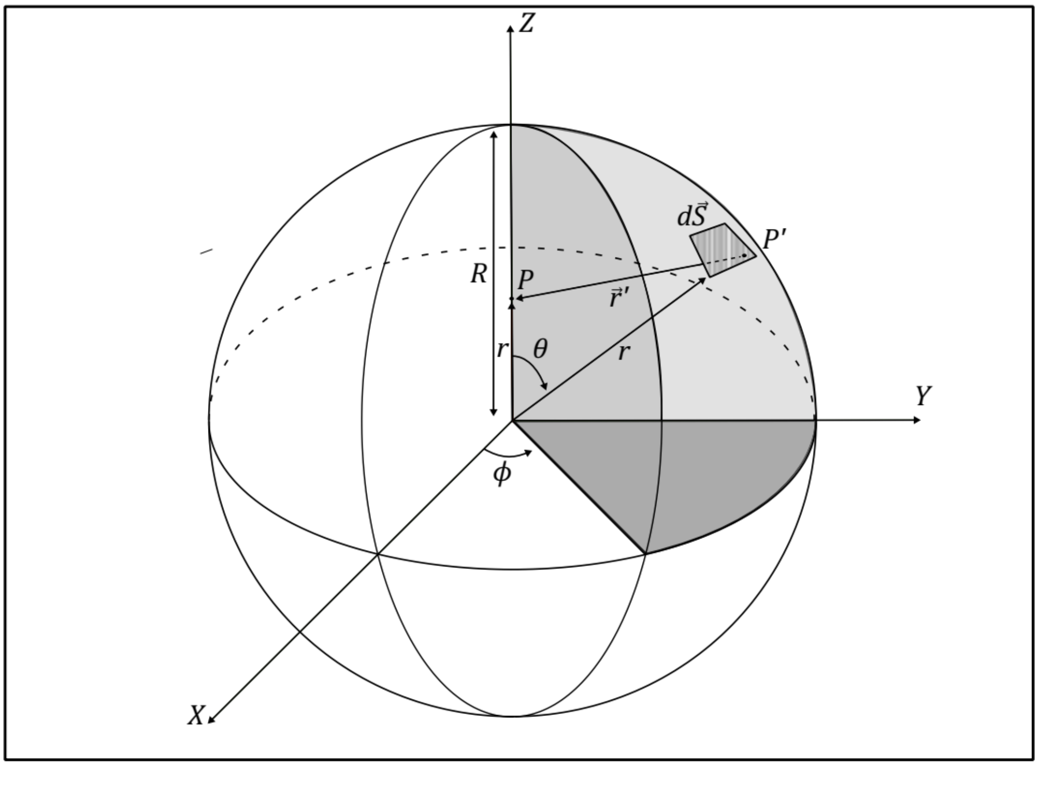

Figure 3. System Representation. Uniformly Charge Sphere Surface of Radius R and Constant Density σ. A Finite Element Around Point P' is Depicted ($d\vec{S}$)

Let consider an arbitrary point (P) inside of the sphere, at a distance $r < R$ to the origin. As the reference system may be arbitrary selected, let consider this point over the Z-axis (Figure 3). On the other hand, consider any point on the charged spherical surface (P'). Let express these two points in a Cartesian basis reference system with its components in spherical coordinates, such as (6).

$$P = (x, y, z) = (0,0,r) = r\hat{k} \tag{6}$$

$$\begin{aligned} P' = (x, y, z) &= (R\sin\theta\cos\phi\,, R\sin\theta\sin\phi\,, R\cos\theta) \\ &= R\sin\theta\cos\phi\,\hat{\imath} + R\sin\theta\sin\phi\,\hat{\jmath} + R\cos\theta\,\hat{k} \end{aligned} \tag{7}$$

Then, the distance between these arbitrary two points is (8).

$$r' = \left|\vec{P} - \vec{P}'\right| = \sqrt{R^2 + r^2 - 2rR\cos\theta} \tag{8}$$

The charge of any surface finite element around P' is, by definition (9).

$$dq = \sigma dS = \sigma R^2 \sin\theta\, d\theta d\phi \tag{9}$$

Thus, the electric potential that finite element on P' creates on arbitrary inner point P, can be approximated by using the known point charge potential equation (10).

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r'} = \frac{1}{4\pi\epsilon_0}\frac{\sigma R^2 \sin\theta\, d\theta d\phi}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} \tag{10}$$

Where ϵ_0 is the permittivity of free space. Remarks that, this equation involves a null potential at infinity as a boundary condition. Applying the superposition principle, the electric potential over P is the sum of all surface finite elements potential, such as (11).

$$V = \int_S \frac{1}{4\pi\epsilon_0}\frac{\sigma R^2 \sin\theta\, d\theta d\phi}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} \tag{11}$$

Where S is the entire sphere surface. Then (12).

$$V = \frac{\sigma R^2}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} d\phi = \frac{\sigma R^2}{2\epsilon_0}\int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} \tag{12}$$

To calculate this integral, the following substitution is used (13).

$$u = cos\theta \Longrightarrow du = -sin\theta d\theta \tag{13}$$

Then,

$$V = -\frac{\sigma R^2}{2\epsilon_0}\int_{u=1}^{u=-1} \frac{du}{\sqrt{R^2 + r^2 - 2rRu}} \tag{14}$$

Which is a direct integral,

$$V = \frac{\sigma R^2}{2\epsilon_0}\left[\frac{\sqrt{R^2 + r^2 - 2rRu}}{rR}\right]_1^{-1} = \frac{\sigma R}{\epsilon_0} \tag{15}$$

Notice that, the electric potential is a constant magnitude inside the sphere for any arbitrary point P. This constant potential may also be written using the total body charge (Q) as (16).

$$V = \frac{\sigma R}{\epsilon_0} = k\frac{Q}{R} \tag{16}$$

Where k is the Coulomb's constant. Then, the electric field inside the sphere ($r < R$) is null, by definition (17).

$$\vec{E} = -\vec{\nabla} V = 0 \tag{17}$$

Finite Ring-Elements Integration

Consider, in this case, the spherical surface as the sum of finite ring-elements along any symmetry axis. Since direction of the reference system is arbitrary, let consider the ring-elements orthogonal to the Z-axis as illustrated at Figure 4. The radius of each ring (R') depends on its distance (z) from the center of the sphere. The ring parameters can be defined in spherical coordinates such as (see Figure 4).

$$\begin{cases} R' = R \cdot sin\theta \\ z = R \cdot cos\theta \end{cases} \tag{18}$$

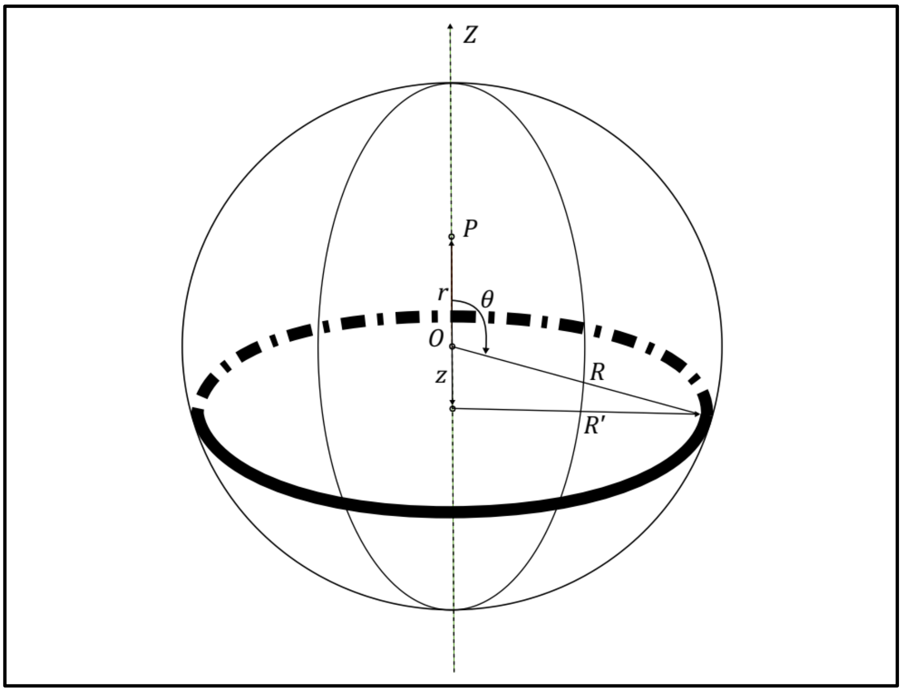

Figure 4. System Representation. Uniformly Charge Sphere Surface of Radius R and Constant Density σ. The Sphere is Composed by Finite Ring-Elements of Radius R' Along the Z −axis. A Finite Element at a Distance z from the Origin is Depicted

The charge associated to each finite ring-element is, by definition (19).

$$dq = \sigma dS = \sigma 2\pi R^2 sin\theta d\theta \tag{19}$$

Being the finite ring-elements width $Rd\theta$. Notice that, finite element charge is maximum over the equator and null over the poles. On the other hand, consider any point P inside the sphere, along the Z-axis, at a distance $r < R$ from the origin of the coordinate system, $P = (0,0,r)$. Notice that, this point is an arbitrary point inside the sphere as the direction selection of the reference system. The point is defined in a Cartesian basis vector space despite the spherical coordinate r is used. The distance from any ring's center to the point P is the next positive parameter (20).

$$|d| = |r - z| = |r - R \cdot cos\theta| \tag{20}$$

The electric field of any ring-element over the P point is, due to the system symmetry, parallel to the Z-axis according to the known equation for a uniformly charge ring.

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq \cdot d}{\left(R'^2 + d^2\right)^{3/2}}\hat{k} \tag{21}$$

At this point, it must be notice that the sign of the factor $d = r - z$ determines the positive-negative direction of the electric field. Furthermore, remarks that the electric field cancels over the point P as it change its direction from one side to the other along this point. Then, substituting the terms and applying the superposition principle yields (22).

$$\vec{E} = \int_S d\vec{E} = \frac{1}{4\pi\epsilon_0}\hat{k}\int_{\theta=0}^{\theta=\pi}\frac{2\pi\sigma R^2 sin\theta(r - Rcos\theta)}{((Rsin\theta)^2 + (r - Rcos\theta)^2)^{\frac{3}{2}}}d\theta \tag{22}$$

Then,

$$\vec{E} = \frac{\sigma R^2}{2\epsilon_0}\hat{k}\int_0^{\pi}\frac{(r - R\cos\theta)\sin\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}}d\theta \tag{23}$$

Again, the next substitution may be used (24).

$$u = cos\,\theta \Rightarrow du = -\,sin\,\theta\,d\theta \tag{24}$$

Thus,

$$\vec{E} = \frac{\sigma R^2}{2\epsilon_0}\hat{k}\int_1^{-1}\frac{(Ru - r)du}{(r^2 + R^2 - 2rRu)^{3/2}} \tag{25}$$

Which is the sum of the next two integrals that cancels themselves,

$$\vec{E} = \frac{\sigma R^2}{2\epsilon_0}\hat{k}\left(R\int_1^{-1}\frac{udu}{(r^2 + R^2 - 2rRu)^{3/2}} - r\int_1^{-1}\frac{du}{(r^2 + R^2 - 2rRu)^{3/2}}\right) = 0 \tag{26}$$

As might be observed ($I_1 = I_2$),

$$\begin{aligned} I_1 = R\int_1^{-1}\frac{udu}{(r^2 + R^2 - 2rRu)^{3/2}} \\ = R\left[\frac{u}{rR\sqrt{r^2 + R^2 - 2rRu}}\right]_1^{-1} - \frac{1}{r}\int_1^{-1}\frac{du}{\sqrt{r^2 + R^2 - 2rRu}} \\ = \frac{1}{r}\left(\frac{-1}{R + r} - \frac{1}{R - r}\right) + \frac{1}{r^2 R}\left[\sqrt{r^2 + R^2 - 2rRu}\right]_1^{-1} \\ = \frac{-2R}{r(R^2 - r^2)} + \frac{2}{rR} \end{aligned}$$

$$I_1 = -\frac{2r}{R(R^2 - r^2)} \tag{27}$$

$$I_2 = r\int_1^{-1} \frac{du}{(r^2 + R^2 - 2rRu)^{3/2}} = \frac{1}{R}\left[\frac{1}{\sqrt{r^2 + R^2 - 2rRu}}\right]_1^{-1} = \frac{1}{R}\left(\frac{1}{R+r} - \frac{1}{R-r}\right)$$

$$I_2 = -\frac{2r}{R(R^2 - r^2)} \tag{28}$$

Where integration by parts was firstly solved at I_1, and the rest are direct integrals. Hence, a null electric field for any point inside the sphere ($r < R$) is also found by this way: $\vec{E} = 0$. Remarks that, via equation (23) it is possible to find the electric field of a uniformly charged spherical cap of height $h = R - r$ over the point P, by evaluating this integral over the interval $[0, \theta_0]$, where $cos\theta_0 = r/R$. In the case of $\theta_0 = \pi/2$ this equation leads to the electric field of an uniformly charged hemispherical surface (del Vigo & Renedo, 2016).

Direct Integration of the Coulomb's Law Equation

Consider a spherical surface with radius R and uniformly distributed total charge Q with a constant density σ. The sphere center is assumed to be the origin of the reference system. Figure 3 represents this system. Let consider an arbitrary point P inside the sphere, at a distance $r < R$ from the origin, to determine the electric field. As the reference system may be arbitrary selected, this point is assumed to be on the Z-axis (see Figure 3). On the other hand, let P' any point on the uniformly charged spherical surface. Expressing these two points in Cartesian reference system with its components in spherical coordinates yields (29).

$$\begin{cases} P = (0,0,r) \\ P' = (x,y,z) = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta) \end{cases} \tag{29}$$

Notice that, this point position is equivalent to was used at equations (6)-(7) in the previous section. Let also consider a superficial finite element around P'. The charge of this element is, by definition, as equation (9).

$$dq = \sigma dS = \sigma R^2 \sin\theta \, d\theta d\phi \tag{30}$$

According to Coulomb's law, the electric field that this charge creates at point P is (point charge approximation).

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{|r'|^2}\hat{r}' \tag{31}$$

Where ϵ_0 is the permittivity of free space, and $\vec{r}'$ is the vector from P' to P.

$$\vec{r}' = \overrightarrow{P'} - \vec{P} = (-R\sin\theta\cos\phi, -R\sin\theta\sin\phi, r - R\cos\theta) \tag{32}$$

Then, unitary vector is (33),

$$\hat{r}' = \frac{\vec{r}'}{|r'|} = \frac{(-R\sin\theta\cos\phi\,, -R\sin\theta\sin\phi\,, r - R\cos\theta)}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} \tag{33}$$

Notice that, Cartesian coordinates system is used as a basis of the vector system, even though the components are expressed in spherical coordinates (34).

$$\vec{r}' = -R\sin\theta\cos\phi\,\hat{\imath} - R\sin\theta\sin\phi\,\hat{\jmath} + (r - R\cos\theta)\hat{k} \tag{34}$$

The advantage of this framework is that, these constant unitary vectors may be extracted from the integrals involved. The total electric field over P can be determined applying the superposition principle for all the sphere finite elements, such as (35).

$$\begin{aligned}\vec{E} &= \int_S d\vec{E} \\ &= \frac{\sigma R^2}{4\pi\epsilon_0}\int_S \frac{\left(-R\sin\theta\cos\phi\,\hat{\imath} - R\sin\theta\sin\phi\,\hat{\jmath} + (r - R\cos\theta)\hat{k}\right)\sin\theta}{(R^2 + r^2 - 2rR\cos\theta)^{3/2}}\,d\theta d\phi\end{aligned} \tag{35}$$

Being S, the total surface sphere. This equation represent the sum of three integrals that may be solved independently, such as (36).

$$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k} \tag{36}$$

First, the x-component of the electric field (37).

$$E_x = -\frac{\sigma R^3}{4\pi\epsilon_0}\int_0^{\pi} \frac{\sin^2\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}}\,d\theta \int_0^{2\pi} \cos\phi\,d\phi = 0 \tag{37}$$

Where separation variable method was applied. This integral is null since,

$$\int_0^{2\pi} \cos\phi\;d\phi = 0 \tag{38}$$

Secondly, the y-component of the electric field is (39).

$$E_y = -\frac{\sigma R^3}{4\pi\epsilon_0}\int_0^{\pi} \frac{sin^2\,\theta}{(r^2 + R^2 - 2rR\,cos\,\theta)^{3/2}}\,d\theta \int_0^{2\pi} sin\,\phi\,d\phi = 0 \tag{39}$$

What is also zero, since (40).

$$\int_0^{2\pi} \sin\phi\,d\phi = 0 \tag{40}$$

And, finally, the z-component of the electric field is (41).

$$E_z = \frac{\sigma R^2}{4\pi\epsilon_0}\int_0^{\pi} \frac{(r - R\,cos\,\theta)\,sin\,\theta}{(r^2 + R^2 - 2rR\,cos\,\theta)^{3/2}}\,d\theta \int_0^{2\pi} d\phi \tag{41}$$

The integral over ϕ is straightforward. Thus arises (42).

$$E_z = \frac{\sigma R^2}{2\epsilon_0}\int_0^{\pi} \frac{(r - R\cos\theta)\sin\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}} d\theta \tag{42}$$

Which is equation (23), omitting the unitary vector, that is null as was proved in the previous section. Then, a null contribution to the electric field is found inside the sphere $(r < R)$ (43).

$$\vec{E} = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{k} = (0,0,0) \tag{43}$$

These findings further reinforce the theoretical and numerical consensus that the electric field inside a uniformly charged spherical surface is zero. For instance, Sheng et al. (2023) provided a rigorous analytic derivation of the electric potential for arbitrary uniformly charged polygons, which includes the spherical case as a special application demonstrating a null interior field. Moreover, Jang et al. (2023) utilized finite element simulations to study electrostatic potential and field distributions—although in a lightning rod context, their methodology validates that symmetric charge configurations produce an interior field effectively approaching zero, paralleling the analytic results.

CONCLUSIONS

Traditional Physics and Electromagnetism textbooks have well established that the electric field vanishes inside a conducting spherical surface in electrostatic equilibrium, what typically justify by using Gauss's law and symmetry arguments. However, the detailed mathematical proof of this result is normally omitted in academic literature despite its instructive importance. The formal proofs of physics formulations are highly interesting in academic context to develop the student's criticism and creative way of think. This is, perhaps, the main purpose of this paper. In this way, it is also pedagogical to derive a result by different mathematical ways or hypothesis, as performed in this article, what moreover, let to confirm the problem solution. Nevertheless, the electric field within this system remains an open question that can only be resolved through experimental investigation. As mentioned before, the absence of an electric field inside a uniformly charged spherical surface is a mathematical consequence derived from the inverse-square dependence of Coulomb's law ($E \propto 1/r^2$), that was also used in the mathematical derivations involved in this article. Then, if Coulomb's law fails to preserve this symmetry it would imply the existence of a weak electric field within the system.

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